Asset Selection and Under-Diversification with Financial Constraints and Income: Implications for Household Portfolio Studies

ABSTRACT

Empirical studies of household portfolios have shown that young and relatively poor households hold under-diversified portfolios that are concentrated in a small number of assets, a fact often attributed to various behavioral biases. We present a model that offers a potential rational alternative: we show that relatively poor investors; i.e., investors with little financial wealth, who receive labor income and have access to multiple risky assets rationally limit the number of assets they invest in when faced with financial constraints such as margin requirements and restrictions on borrowing. We provide both theoretical and numerical support for our results and show, in an example calibrated to returns of five industry portfolios from 1927 to 2004, that while older investors optimally hold diversified portfolios, younger investors with labor income prefer portfolios that are concentrated in high-tech stocks that offer higher expected returns. Our results suggest that the ratio of financial wealth to labor income would be a useful control variable in household portfolio studies.

JEL Classification: D81, D83, E21, G11

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Introduction

Portfolio choice has been a topic studied extensively in the literature. Starting at least as early as Merton (1971) and Cass and Stiglitz (1970), theory suggests that the equity part of any investor's portfolio should include all the risky assets available held in the same proportions, with the mix between bonds and stocks determined by individual risk aversion. The prescription, often called *mutual fund separation theorem*, has partly been the reason for the explosive growth in the size of mutual funds that track the market portfolio over the last 40 years.

More recently, researchers have been able to empirically study household portfolios and have found deviations from the theoretical prescription: household portfolios are under-diversified and concentrated on a small number of stocks. As a sample of the empirical literature, Kelly (1995) studies the 1983 Survey of Consumer Finances and finds that diversification increases with portfolio size, investor age, and investor wealth. Polkovnichenko (2005) uses the 1983, 1989, 1992, 1995, 1998 and 2001 Survey of Consumer Finances and confirms that wealthier households hold more diversified portfolios, even though not all wealthy households are well diversified. He argues that investors are aware of the higher risk associated with undiversified portfolios and proposes preferences with rank dependency as a potential explanation. Ivković, Sialm, and Weisbenner (2008) use data from trades and monthly portfolio positions of retail investors at a large U.S. discount brokerage house for the 1991-1996 period and show that the number of stocks in the portfolio increases with the size of the account balance, and that concentrated portfolios have higher levels of risk and return and lower Sharpe ratios than diversified portfolios. Goetzmann and Kumar (2008) study the same dataset and find that diversification increases with age and income, while households with only a retirement account hold less diversified portfolios than households with additional non-retirement investment accounts. They examine several potential explanations for the lack of diversification: small portfolio size and transaction costs; search and learning costs; investor demographics and financial sophistication; layered portfolio structure; preference to higher order moments; and behavioral biases such as illusion of control, investor over-confidence, local bias and trend-following behavior. Kumar (2009), using the same dataset, finds that young investors have a strong preference for riskier stocks, and argues that the young are more likely to be heavy lottery players, and this is reflected in their selection of stocks. Mitton and Vorking (2007), using a dataset of 60,000 individual accounts find that investors hold underdiversified portfolios with positively skewed returns, a fact they attribute to heterogeneous preferences for skewness. In a recent paper, Calvet, Campbell, and Sodini (2008) study a dataset of the portfolios of the entire Swedish population and propose several measures to quantify the under-diversification of household portfolios. They show that increasing age, wealth and financial sophistication increase diversification, but also lead to investors taking more aggressive positions.

While behavioral biases have been considered the cause of the discrepancy between the theory and many of the empirical observations, in this paper we offer a rational alternative explanation. We extend the theoretical literature by considering an investor who is able to invest in multiple risky assets and who receives labor income and faces financial constraints in the form of margin requirements and borrowing restrictions: investment needs and margin requirements can be satisfied only out of the current financial wealth of the investor, effectively rendering future earnings non-tradable.[1] Our main theoretical result is that investors facing financial constraints do not follow the theoretical prescription described earlier: rather than holding a diversified equity portfolio they optimally choose to concentrate their portfolio on a few assets. The extent to which investors limit their investments is captured by the ratio of their financial wealth to their income. As the ratio of financial wealth to income decreases, investors restrict their equity portfolios, dropping an asset each time a threshold in the value of the financial wealth to income ratio is crossed. We show that in the limit when the financial wealth to income ratio tends to zero, investors' portfolios include a single risky asset, whose choice is entirely based on its leveraged expected return without regard to the asset's volatility or Sharpe ratio.

Our result can be intuitively understood as the combination of two effects: the increased demand for equity exposure when labor income is large compared to financial wealth; and the limited ability to satisfy this demand because of the margin requirements and borrowing constraint. It is simplest to explain the intuition in the case of an investor with deterministic labor income. In this case labor income can be thought of as a fixed investment in a risk-free bond. As shown in Merton (1971), to find the optimal allocation, the investor should discount the expected lifetime income, add it to financial wealth, and choose an equity allocation based on the sum. Keeping financial wealth constant and increasing labor income implies increasing equity investment when measured as a fraction of financial wealth.

[1] Having such restrictions impacts the investor's choices significantly. The restrictions can be attributed to adverse selection and moral hazard problems, as well as the inalienability of human capital. The cause of the constraints is beyond the scope of this paper and we will consider the restrictions as given.

In the absence of the margin requirements and the borrowing constraint the investor would borrow against his future income and increase expected return by leveraging his portfolio while keeping it diversified. Since margin requirements limit the extent that the portfolio can be leveraged, the only possible way to increase expected return is through shifting portfolio holdings towards assets with higher expected returns, sacrificing diversification. We show that as the demand for additional equity exposure increases; i.e., when the financial wealth to income ratio decreases, the demand for higher expected returns prevails. In the limit the investor holds a single risky asset based entirely on the asset's expected return.

An alternative interpretation of our selection result, where the investor trades diversification for higher expected returns is that the investor can be thought of as becoming less risk averse the more the constraints bind. We show that in the limit when the ratio of his financial wealth to income ratio tends to zero the investor acts as if he were risk-neutral, basing his investment choice on expected returns alone. Additionally, investor choices can be understood through adjusted asset characteristics: in the case of deterministic income, using duality, we show that the behavior of an investor that faces financial constraints corresponds to the behavior of an unconstrained investor whose opportunity set includes assets with adjusted Sharpe ratios. As the constraint binds the adjusted Sharpe ratios of the assets decrease; at the point where an asset is dropped from the portfolio, its adjusted Sharpe ratio drops to zero.

We point out that while the investor chooses progressively less diversified portfolios as his financial wealth to income ratio decreases, his overall risk, measured by the variance of returns of his overall portfolio, does not necessarily increase. Since lower values of the financial wealth to income ratio increase demand for equity, the unconstrained investor would have chosen a leveraged portfolio, trading additional variance for additional expected return. Unable to satisfy his risk appetite through leverage because of the constraints, the investor chooses a portfolio that achieves higher expected returns but is less diversified. Whether the variance of the returns of the overall portfolio of the constrained investor is greater or smaller than that of the portfolio of the unconstrained investor hinges on the balance of the two effects.

In addition to changes in the asset allocation, we show that the financial constraints induce changes in the investor's consumption behavior. In the case of an infinitely lived investor that receives an uninterrupted income stream, when the wealth of the investor tends to zero the investor's consumption rate tends to his income rate, preventing wealth accumulation. While this result depends on the assumption of infinite horizon, it does suggest that investors with relatively long horizons, little wealth, and large income, have little incentive to save, a result that we document in our numerical study. Another effect of the margin requirements and borrowing constraint is that investor consumption decreases, and, when income is deterministic, the volatility of the investor's consumption is lower than the volatility of consumption for a similar investor that does not face these financial constraints. The intuition behind this result is that the constrained investor leverages his portfolio less and therefore his overall wealth is less influenced by changes in the prices of the risky assets.

While our theoretical underdiversification results are obtained for a base case of an infinitely-lived investor with constant relative risk aversion preferences that receives income spanned by the risky assets, we demonstrate their robustness by considering several extensions. We show that underdiversification persists when uncertainty in income is not spanned by the risky assets; when the investor has general preferences and receives deterministic income; and also in the case of an investor with finite horizon.[2]

To the best of our knowledge, our paper is the first to link the combination of income and financial constraints to underdiversification of investor portfolios. Literature related to our paper includes the papers by Karatzas, Lehoczky, and Shreve (1987), and Cox and Huang (1989) who introduce martingale techniques that make it easier to deal with constraints on the investment strategies. Models with constraints on the portfolio policies are studied by Karatzas, Lehoczky, Shreve, and Xu (1991), Cvitanić and Karatzas (1992), Cvitanić and Karatzas (1993), He and Pearson (1991), Xu and Shreve (1992) and Tepla (2000). Cuoco (1997) is able to demonstrate existence of optimal strategies for the case of an investor that faces margin constraints and receives income but does not provide a characterization of the strategies. Koo (1998) and Koo (1999) solves the optimal investment and consumption problem for an investor that receives labor income and faces a short-sale constraint and describes properties of the optimal consumption plan, but does not discuss underdiversification. Cuoco and Liu (2000) discuss the case of an investor that is facing margin requirements but does not receive income, and provide a characterization of his optimal investment strategy. He and Pagès (1993), El Karoui and Jeanblanc-Picqué

[2] In the case of an investor with finite horizon, the thresholds where assets are dropped from the investor's portfolio depend on the time remaining until the horizon.

(1998) and Duffie, Fleming, Soner, and Zariphopoulou (1997) study the optimal asset selection problem of an investor who receives income and who is constrained to maintain non-negative levels of current wealth, but do not address margin requirements. Cuoco and Liu (2004) find underdiversification results in a study of the impact of VaR reporting rules in the portfolio choice of a financial institution that maximizes utility from terminal wealth. While very different, the setting in Cuoco and Liu (2004) can be shown to be a special case of our study, corresponding to an infinitely lived investor with constant relative risk aversion preferences that receives deterministic income.[3] Liu (2009) describes a model where investors engage in asset selection due to the combination of a desire to guarantee a minimum level of wealth and constraints on their ability to borrow and to short-sell risky assets. When the level of wealth drops close to the minimum, investors reduce the risk of their portfolios by removing assets from the portfolio. Unlike our paper, the variable that determines whether assets are included in the investor's portfolio, is the investor's financial wealth rather than the ratio of his financial wealth to his income. While in our setting we are able to predict a relationship between the degree of underdiversification of an investor's portfolio and the investor's age, it is unclear what the impact of a required minimum level of wealth would be in an investment and consumption problem over the lifetime of an investor; i.e., whether a required minimum level of wealth would lead younger investors to hold an underdiversified portfolio.

To quantify the magnitude of our theoretical results, we also present a numerical algorithm to solve a lifecycle problem of optimal consumption and asset allocation in the presence of constraints. We apply the algorithm in the case of an investor that receives income until age 65 and then retires with an expected remaining lifetime of 20 years. The investor has access to five risky assets, calibrated to match the risk-return characteristics of five industry portfolios based on data from 1927-2004. The algorithm, originally introduced in Yang (2009), is an extension of the algorithm developed by Brandt, Goyal, Santa-Clara, and Stroud (2005).[4] The algorithm determines optimal asset allocations by solving the first order Karush-Kuhn-Tucker conditions using functional approximation of conditional expectations and projection of the value function on a set of radial basis functions to address the curse of dimensionality problem when facing a large number of state and choice variables. The extension allows for

[3]Cuoco and Liu (2004) do not study the problem of optimal consumption and asset allocation for an investor that faces financial constraints and receives labor income — rather they consider the problem of asset allocation for a financial institution that faces a VaR constraint.

[4]See also Carroll (2006), and Garlappi and Skoulakis (2008).

a more accurate estimation of conditional expectations by limiting the region where test solutions are generated iteratively, a process called "Test Region Iterative Contraction (TRIC)". Our numerical results are in line with the theoretical intuition: young investors hold under-diversified portfolios, engage in asset selection, and save a smaller fraction of their income compared to older investors, who hold portfolios that are close to diversified. It is interesting to note that our calibration implies that when investors are severely constrained they only choose high-tech stocks for their equity portfolio, increasing the expected return of their portfolio but lowering its Sharpe ratio. We provide values for the under-diversification measures developed by Calvet, Campbell, and Sodini (2008) and show that, in line with our theoretical result, the investor's effective risk aversion tends to zero when the value of the financial wealth to income ratio tends to zero.

The remainder of the paper is organized as follows: in Section I we present our theoretical model and results. Section II discusses the numerical algorithm used to solve the finite horizon problem, and presents the numerical results for the optimal allocations and diversification measures for our calibrated model of five industry indices. Section III concludes. The proofs are contained in the online Appendix.

I. Theoretical Analysis

A. The Economic Setting

For our theoretical results, we consider a continuous time economic setting with an infinitely lived investor who derives utility from consumption and who is able to invest in a riskless money-market account and N risky securities that evolve according to geometric Brownian motion with constant coefficients.[5] We assume that the investor's utility is of the constant relative risk aversion (CRRA) type.[6]

The Financial Market.

Uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, P)$ on which an N dimensional, standard, Brownian motion $w = (w_1, w_2, ..., w_N)$ is defined. A state of nature ω is an element of Ω. $\mathcal{F}$ denotes

[5]We have chosen geometric Brownian motion for tractability reasons.

[6]Our results also hold for general preferences when the labor income received by the investor is deterministic.

the tribe of subsets of Ω that are events over which the probability measure P is assigned. At time t, the investor's information set is $\mathcal{F}_t$, where $\mathcal{F}_t$ is the σ-algebra generated by the observations of w, $\{w_s; 0 \leq s \leq t\}$. The filtration $\mathbb{F} = \{\mathcal{F}_t, t \in \mathbb{R}_+\}$ is the information structure and satisfies the usual conditions (increasing, right-continuous). In our setting only a money-market account that pays a constant interest rate and N risky securities are available. The value of the money-market account, B evolves according to

$$dB_t = rB_t dt, \tag{1}$$

where r is the constant interest rate. Let $S = (S_1, S_{2,} ..., S_N)$ be the vector stock price process whose dynamics are given by

$$dS_t = I_{S_t}\mu dt + I_{S_t}\sigma dw_t, \tag{2}$$

where I_S is a diagonal $N \times N$ matrix with diagonal elements S, μ is an $N \times 1$ matrix, $\sigma = [\sigma_{ij}]$ is an $N \times N$ matrix and dw_t is the increment of an N dimensional Wiener process $w = (w_1, w_2, ..., w_N)$ with $[dw_{it}, dw_{jt}] = \delta_{ij}$ where $\delta_{ij} = 0, i \neq j$ and $\delta_{ii} = 1$. The instantaneous covariance matrix $\sigma\sigma^\intercal$ is assumed to be non-singular.

Trading Strategies and Margin Requirements.

We assume that consumption c_t and trading strategies (x_t, z_t) are adapted processes to the filtration $\mathbb{F}$, where x is the dollar amount invested in the money-market account and $z^\intercal = (z_1, z_2, ..., z_N)$, are the dollar amounts invested in the N risky assets.

To trade in risky assets, U.S. investors must hold sufficient wealth in a margin account. This wealth can be held in securities or cash. The Federal Reserve Board's Regulation T sets the initial margin requirement for stock positions undertaken through brokers. The values for the initial margin requirement are 50 percent for a long equity position, and 150 percent for a short equity position.[7]

[7] See Fortune (2000) as well as the Federal Reserve Board's Regulation T for institutional details. In addition to initial margin requirements, there are also maintenance margin requirements that correspond to the level in the margin account at which collateral needs to be added to the account to avoid liquidation of the position. Including a maintenance margin would make the problem path dependent and we do not consider it in this paper. Further complications regarding margin accounts include the fact that 102% of the collateral held in the margin account needs to be held in liquid assets, and that the collateral does not, in general, earn the risk free rate of interest, see Geczy, Musto, and Reed (2002). While it would be interesting to consider these additional features, our main result regarding underdiversification of an investor's portfolio when the investor's labor income is large compared to his financial wealth, would not change.

For our model, we impose the following margin constraint on an investor that holds $z_i, i = 1,2,...,N$ dollar amounts in the risky assets

$$\lambda^+(z^+)^\intercal\overline{1} + \lambda^-(z^-)^\intercal\overline{1} \leq W, \tag{3}$$

with $\overline{1}^\intercal = (1,1,...,1) \in \mathbb{R}^N$, and $\lambda^+ = 1-\kappa^+, \lambda^- = \kappa^- - 1$ with $0 \leq \kappa^+ \leq 1, 1 \leq \kappa^-$.[8] The regulation T initial margin requirements correspond to $\kappa^+ = 0.5$ and $\kappa^- = 1.5$.

We define the set of all possible margin coefficients, Λ:

$$\Lambda = \{\lambda \in \mathbb{R}^N,\ \lambda_i \in \{\lambda^+, -\lambda^-\},\ i = 1,...,N\},$$

and the set of all feasible allocations in the risky assets, Q:

$$Q = \{x \in \mathbb{R}^N, \lambda^\intercal x \leq 1,\ \lambda \in \Lambda\}.$$

Q is a convex set prescribed by 2^N linear constraints of which at most N are binding at the same time. Risky investment z satisfies the margin constraint (3) if and only if z/W is in Q. We note that the margin constraint is more stringent than the constraint of non-negative wealth $W \geq 0$.

Income. We assume that the investor receives a non-negative income stream at a rate Y_t, which may be stochastic and is spanned by the risky assets in the economy.

$$dY_t = Y_t\,(mdt + \Sigma^\intercal dw_t) \tag{4}$$

where m is the growth rate of income, and $\Sigma^\intercal = (\Sigma_1, \Sigma_2, .., \Sigma_N) \in \mathbb{R}^N$. All the coefficients are assumed to be constant.

Preferences.

[8]For any real number x, we have $x = x^+ - x^-$, with $x^+ = \max(x,0)$ and $x^- = \max(-x,0)$.

There is a single perishable good available for consumption, the numéraire. Preferences are represented by a time additive utility function

$$U(c) = E\left[\int_0^\infty u(c_t)e^{-\theta t}dt\right],$$

where the time discount factor, θ, is constant. The utility function u is of the CRRA type, with risk aversion coefficient γ

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma}, & \gamma \neq 1 \\ \ln c, & \gamma = 1 \end{cases}$$

Optimization Problem. The investor's problem is to maximize his expected, cumulative, discounted utility of consumption

$$F(W_t, Y_t) = \max_{(c,\ \frac{z}{W} \in Q)} E_t\left[\int_t^\infty u(c_s)e^{-\theta(s-t)}ds\right], \tag{5}$$

under the budget constraint

$$dW_s = \left(rW_s - c_s + Y_s + z_s^\intercal(\mu - r\overline{1})\right)ds + \sigma z_s^\intercal dw_s,$$

where $W_t > 0, Y_t > 0$ are the initial conditions for the investor's wealth and income rate.

Transversality Condition.

The transversality condition for this problem is given by

$$\lim_{T\to\infty} E_t\left[F(W_{t+T}, Y_{t+T})e^{-\theta(T+t)}\right] = 0.$$

Properties of the Primal Value Function F.

The value function F satisfies the following propositions whose proof can be found in the online Appendix.

Proposition I.1. *F is homogenous of degree $1-\gamma$ in (W,Y).*

Proposition I.2. *F is non-decreasing in W and Y and jointly concave in (W,Y).*

The homogeneity of F allows us to rewrite the value function in terms of the ratio of wealth over income, as

$$F(W,Y) = Y^{1-\gamma} f(\frac{W}{Y}),$$

for some non-decreasing smooth function f. We will refer to f as the reduced value function. Below, we denote by $v = W/Y$ the ratio of wealth over income. Notice that the concavity of F in (W,Y) implies that f is concave in v.

Conditions on Parameters.

In order to get a well defined problem, we impose that the following parameters A, B, C are positive. These parameters appear in the characterization of the value function, the optimal consumption plan and the optimal investment strategy in both the optimization problem of an unconstrained as well as that of a constrained investor

$$\begin{aligned}
A^{-1} &= \frac{\theta}{\gamma} + \frac{\gamma-1}{\gamma}\left(r + \frac{1}{2\gamma}(\mu - r\overline{1})^{\intercal}(\sigma\sigma^{\intercal})^{-1}(\mu - r\overline{1})\right) > 0 \\
B^{-1} &= r - m + (\sigma\Sigma)^{\intercal}(\sigma\sigma^{\intercal})^{-1}(\mu - r\overline{1}) > 0 \\
C^{-1} &= \theta + (\gamma-1)\left(m - \gamma\frac{\Sigma^{\intercal}\Sigma}{2}\right) > 0.
\end{aligned} \tag{6}$$

Dropping the time index t, the Hamilton-Jacobi-Bellman equation for the reduced value function f is given by

$$\begin{aligned}
\left(\theta + (\gamma-1)(m - \gamma\frac{\Sigma^{\intercal}\Sigma}{2})\right) f(v) = &\max_{\omega\in Q} \frac{\gamma(f'(v))^{\frac{\gamma-1}{\gamma}}}{1-\gamma} + f'(v) + (r - m + \gamma\Sigma^{\intercal}\Sigma)\, v f'(v) + \frac{\Sigma^{\intercal}\Sigma}{2} v^2 f''(v) \\
&+ \omega^{\intercal}\left((\mu - r\overline{1} - \gamma\sigma\Sigma) v f'(v) - \sigma\Sigma v^2 f''(v)\right) + v^2 f''(v)\omega^{\top}\frac{\sigma\sigma^{\intercal}}{2}\omega,
\end{aligned}$$

where $\omega = z/W$ is the vector of the percentage of wealth invested in each of the risky assets. This problem is equivalent to

$$\max_{\omega\in Q}\left[\omega^{\intercal}(\mu - r\overline{1} + (y-\gamma)\sigma\Sigma) - \frac{y}{2}\omega^{\intercal}\sigma\sigma^{\intercal}\omega\right], \tag{7}$$

where y is the investor's lifetime relative risk aversion, $y = -WF_{11}/F_1$. As shown in the online Appendix, the boundary condition at $v = W/Y = 0$, is given by

$$\frac{1}{f(0)} = (1-\gamma)\left(\theta + (\gamma-1)(m - \gamma\frac{\Sigma^{\intercal}\Sigma}{2})\right).$$

B. Benchmark Case: No Margin Requirements

Following Merton (1971), when the investor does not face margin requirements markets are complete. The optimal asset allocations z^f, and optimal consumption c^f are given by

$$\begin{aligned} z^f &= \frac{(\sigma\sigma^{\intercal})^{-1}\left(\mu - r\overline{1}\right)}{\gamma} W - B\frac{(\sigma\sigma^{\intercal})^{-1}\eta}{\gamma} Y \\ c^f &= \frac{W + BY}{A}, \end{aligned}$$

where $\eta = \mu - r\overline{1} - \gamma\sigma\Sigma$ is the vector of expected excess returns adjusted for labor income correlation with the risky assets.

C. Case with Margin Requirements

Cuoco and Liu (2000) characterize the optimal consumption and portfolio choices for an investor that is subject to margin requirements and that does not receive income; i.e., $Y \equiv 0$. The case of an investor that receives an income stream and is subject to margin requirements is considerably more complicated than the case without income. Intuitively, from the work of Merton (1971), we know that without margin requirements the investor should discount his future earnings, add the discounted value to his current wealth, and make an investment choice based on the sum, provided he can borrow against his future earnings. Since discounted future earnings may be a significant portion of the sum, and possibly many times the current wealth, the allocation may violate the margin constraint. The extent to which the margin constraint binds depends on the ratio between the current wealth and the discounted value of future earnings.

Before addressing the general case, we first consider the case where the adjusted excess return vector η is identically equal to zero. In this case, we show in the online Appendix that labor income has no impact on portfolio holdings and the fraction of wealth invested in each asset is constant. If $(\sigma\sigma^{\intercal})^{-1}\sigma\Sigma \in Q$, the margin constraint is never binding and $z^*/W = (\sigma\sigma^{\intercal})^{-1}\sigma\Sigma$. Otherwise, the margin constraint is always binding. Depending on the parameters of the model, the number of assets in the portfolio can range from N (full diversification) to 1 (full selection). A condition under which exactly K assets are optimally held in the portfolio is reported in the online Appendix.

In the remainder of the paper we assume that the adjusted excess return vector is not equal to zero, $\eta \neq 0$. To study the impact of the income stream on the asset allocations, we choose parameters such that the margin requirement is not binding for low income levels.[9] This assumption is satisfied if the following inequality holds

$$\max_{\lambda\in\Lambda}\left(\frac{\lambda^{\intercal}(\sigma\sigma^{\intercal})^{-1}\eta}{\gamma} + \lambda^{\intercal}(\sigma\sigma^{\intercal})^{-1}\sigma\Sigma\right) < 1. \tag{8}$$

Proposition I.3 below characterizes the optimal portfolio allocations according to the investor's lifetime relative risk aversion y, or, equivalently, in terms of the ratio, $v = W/Y$, of current financial wealth to income. In order to state the proposition, we define $e_i^{\intercal} = (0,0,...,1,...,0)$, where 1 is in the ith position, and I_K to be the $K \times N$ matrix that consists of the first K rows of the $N \times N$ identity matrix.

Proposition I.3. *Given the evolution of the price of the risky assets, money-market account, and income, described by equations (1), (2), and (4), and under the assumptions on the parameters given by equations (6), and assuming that the vector of adjusted excess returns,* η, *is not equal to zero, the optimal asset allocation can be described in terms of* $N+1$ *distinct regions, defined by the values of the lifetime relative risk aversion* y, *or, equivalently the values of the current wealth to income ratio* v. *We characterize these* $N+1$ *regions in terms of the thresholds* $y_{i,i}, i = 1,\dots,N+1$, *defined below. Let*

$$y_{N+1,N+1} = \max_{y}\{0 < y < \gamma,\ \max_{\lambda\in\Lambda}\frac{\lambda^{\intercal}(\sigma\sigma^{\intercal})^{-1}}{y}(\eta + y\sigma\Sigma) \geq 1\}.$$

Then, the threshold value $y_{N+1,N+1}$ *exists, is unique, and*

$$y_{N+1,N+1} = \frac{(\lambda^*)^{\intercal}(\sigma\sigma^{\intercal})^{-1}\eta}{1-(\lambda^*)^{\intercal}(\sigma\sigma^{\intercal})^{-1}\sigma\Sigma},$$

[9] Our motivation behind this assumption is that is that it generates dynamics for the investor's asset allocation that are more complicated compared to the case when the margin requirement binds for low, or zero, levels of income.

for some $\lambda^* \in \Lambda$. *Next, for* $K \in \{1,2,\ldots,N\}, i \in \{1,2,\ldots,K\}$ *let*

$$y_{i,K} = \frac{\left(\lambda^* - \frac{(I_K\lambda^*)^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^*}{e_i^\intercal I_K^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^*} e_i\right)^\intercal I_K^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\eta}{1 - \left(\lambda^* - \frac{(I_K\lambda^*)^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^*}{e_i^\intercal I_K^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^*} e_i\right)^\intercal I_K^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\sigma\Sigma}.$$

and, without loss of generality, assume that risky assets can be ordered such that

$$y_{K,K} = \max_{i=1,..,K} \{y_{i,K},\ 0 < y_{i,K} < y_{K+1,K+1}\},$$

with $y_{K,K} > 0$, $K = 2,\ldots N$. *Then, we have*

$$0 = y_{1,1} < y_{2,2} < y_{3,3} < \ldots < y_{N,N} < y_{N+1,N+1}.$$

When the current wealth to income ratio v is large enough such that the lifetime relative risk aversion y is greater than $y_{N+1,N+1}, y > y_{N+1,N+1}$, *the margin constraint is not binding and the investor holds all N risky assets in his portfolio. As the ratio v decreases, y decreases and at* $y = y_{N+1,N+1}$, *the margin constraint starts binding; margin requirement coefficient* λ_i^* *is determined by whether the position in asset i is long or short at* $y = y_{N+1,N+1}$

$$\lambda_i^* = \begin{cases} \lambda^+, & \text{if } e_i^\intercal(\sigma\sigma^\intercal)^{-1}(\eta + y_{N+1,N+1}\sigma\Sigma) > 0 \\ \lambda^-, & \text{if } e_i^\intercal(\sigma\sigma^\intercal)^{-1}(\eta + y_{N+1,N+1}\sigma\Sigma) < 0 \end{cases}, i = 1,\ldots,N \tag{9}$$

The second region is delimited by values of the current wealth to income ratio v such that

$$y_{N,N} < y < y_{N+1,N+1}.$$

In this region the margin constraint is binding and all assets are held in the portfolio. The third region is delimited by values of v such that

$$y_{N-1,N-1} < y < y_{N,N},$$

In this region the constraint becomes more binding and asset N is optimally dropped out of the portfolio. In general, as the relative risk aversion y decreases, the constraint becomes more binding and, for

$$y_{K,K} < y < y_{K+1,K+1},$$

exactly the first K securities are optimally held; the last $N-K$ securities have sequentially been dropped out of the portfolio. Finally, the last region corresponds to small values of the current wealth to income ratio v and is defined by

$$0 = y_{1,1} < y < y_{2,2}.$$

In this region, the investor holds only one asset to the maximum extent allowed by the margin requirement.

The proof of Proposition I.3 is provided in the online Appendix. We provide the intution behind the proof in the next subsection, in the case of deterministic income.

Proposition I.3 indicates that the investor engages in asset substitution as the margin constraint becomes binding. Intuitively, the investor tries to improve his return, within the bounds of the margin requirement imposed on him, and, in doing so, shifts his portfolio composition toward fewer assets.[10] An alternative intuition can be described in terms of the investor's lifetime relative risk aversion y: lifetime relative risk aversion is high when income is relatively small compared to the investor's wealth, while it is low when discounted future earnings are much larger than current wealth. When lifetime relative risk aversion is low, the investor is willing to increase his exposure in risky assets, resulting in the margin constraint binding and leading the investor to hold fewer assets. In the limit, when the investor's current wealth is negligible compared to his future earnings, the investor acts as if he is risk-neutral: he holds a single risky asset to the maximum extent allowed by the margin constraint, chosen solely based on the asset's expected excess return adjusted for labor income correlation $\mu_1 - r - \gamma\sigma_1\Sigma_1$.

[10]Cuoco and Liu (2004) find risk shifting behavior similar to the one we describe in this paper in the context of a financial institution that needs to follow VaR reporting rules and that tries to optimize its asset selection. This behavior leads the financial institution to invest in under-diversified portfolios as the VaR constraint becomes binding. Through a transformation, their setting can be shown to be a special case of ours, corresponding to the case of deterministic income.

D. Deterministic Income

In the special case where income is deterministic we are able to obtain a result stronger than the result in Proposition I.3. In the online Appendix we show that in this case Proposition I.3 holds beyond the case of investors with CRRA preferences: asset selection occurs as the lifetime relative risk aversion varies for preferences represented by any smooth and concave utility function. Moreover, in the first, non-binding, region the two fund separation theorem applies with the risky assets being held in the same proportions as in the unconstrained case. However, the investor's risky asset allocation is smaller (in absolute value) than the allocation of an investor with the same wealth and income that does not face a margin requirement.

The intuition behind Proposition I.3 is illustrated for the case of deterministic income in Figure 1. The figure considers an investor that has access to three risky assets. The parameters are chosen such that when financial wealth is very large relative to income the investor holds a portfolio that includes all three risky assets. A similar investor with relatively less financial wealth increases his allocation to the risky assets. The larger income is, relative to financial wealth, the higher a percentage of financial wealth the investor places in risky assets. As long as the constraint is not binding the investor maintains the relative proportions of the risky assets in his equity portfolio, keeping the portfolio diversified. At some point, for an investor with low enough financial wealth relative to his income, the size of the equity portfolio is large enough for the margin constraint to bind. In the figure this happens when the chosen allocation reaches the shaded plane. For an investor with even lower financial wealth to income ratio the margin constraint restricts him in choosing portfolios on the shaded plane in Figure 1. As long as the allocation does not reach the edge of the plane the investor maintains all the risky assets in his portfolio but in proportions that vary with the ratio of financial wealth to income. Once the allocation reaches an edge of the shaded plane the investor is restricted to choose portfolios along that edge, and is no longer holding all the risky assets. Further decreases in the financial wealth to income ratio eventually lead the investor to hold an equity portfolio consisting of a single asset, represented in the figure by a vertex. Given our results, in the case of deterministic income, the asset eventually held is the one with the highest expected return, irrespective of the asset's volatility.

If, in addition to deterministic income, we assume that the returns of the risky assets are independent of each other, we can characterize investor behavior still further. In this case the excess return $\mu_k - r$ of risky asset k, and the corresponding margin requirement coefficient λ_k must have the same sign, and risky securities can be ranked according to their leveraged excess expected return

$$0 < \frac{\mu_N - r}{\lambda_N} < \frac{\mu_{N-1} - r}{\lambda_{N-1}} < \cdots < \frac{\mu_1 - r}{\lambda_1}.$$

Asset N is the first security to be dropped out of the portfolio, followed by asset $N-1$ and so on until finally only asset 1 remains in the portfolio.

Relying on duality techniques developed by Cvitanić and Karatzas (1992) and Cuoco (1997), it is possible to interpret the asset allocation and consumption problem with margin requirements as an intertemporal consumption-investment problem for an investor facing no financial constraints by adjusting the risky assets' returns and the risk-free interest rate. This approach can be used to quantify the impact of the margin requirements: as the margin constraint becomes more stringent, adjusted Sharpe ratios for the risky assets shrink (in absolute value), which makes risky assets less attractive to the investor. We provide further details in the online Appendix.

E. Properties of the Consumption and Investment Plans

In addition to our result on asset selection, we can characterize optimal consumption using the following propositions, whose proofs are provided in the online Appendix.

Proposition I.4. *The optimal consumption is increasing in current wealth and current income and is lower than its unconstrained counterpart. Inside the non-binding region, z_k^*/W the fraction of wealth invested in risky asset k is lower (higher) than its unconstrained counterpart z_k^f/W whenever $e_k^{\intercal}(\sigma\sigma^{\top})^{-1}\eta \geq 0 (\leq 0)$. Under the same condition, it is increasing (decreasing) in income (wealth).*

Proposition I.5. *In the limit of zero current wealth, the lifetime relative risk aversion y goes to zero and the optimal consumption rate is equal to the income rate.*

We note that Proposition I.5 implies that an investor with zero current wealth will never be able to accumulate wealth and will always consume his income. This result indicates the extent to which the margin constraint renders holding risky assets unattractive. While the result holds for the infinite horizon setting, in the numerical section we also consider the life-cycle problem, where the investor receives income for only part of his life and then retires and consumes his accumulated wealth.

Proposition I.6. *When income is deterministic, for any level of consumption c, the optimal consumption process has a lower volatility than its unconstrained counterpart. Furthermore, as the margin constraint becomes more binding, consumption volatility decreases down to zero. These results hold for every strictly concave utility function.*

The intuition behind Proposition I.6 is that a margin-constrained investor that receives deterministic income faces less uncertainty than a similar investor that does not face a margin requirement: even though the constrained investor holds an under-diversified portfolio, the magnitude of the portfolio is relatively small compared to the portfolio of the unconstrained investor. Given the smaller portfolio size, random fluctuations in the stock prices have a smaller impact on the sum of the investor's wealth and discounted future earnings, resulting in a smoother consumption pattern. We point out the result does not necessarily hold in the case of stochastic income: if the stochastic income is highly correlated with the risky assets, investment in the risky assets can act as a hedge, smoothing out income shocks, rather than magnifying asset price shocks due to the larger size of the investment portfolio.

F. Unspanned Stochastic Income

Under the financial market described above Duffie, Fleming, Soner, and Zariphopoulou (1997) study the Merton problem for a HARA preference investor who receives labor income that follows geometric Brownian motion that is not perfectly correlated with the stock market. A formal analysis of the existence and uniqueness of the solution of this problem under margin requirements is beyond the scope of our paper. However, a heuristic derivation of the Hamilton-Jacobi-Bellman equation can provide some insight regarding portfolio selection. If the dynamics of labor income Y are given by

$$dY_t = Y_t\left(mdt + \Sigma^\intercal dw_t + \Theta^\intercal dw_t^Y\right),$$

where $\Theta^\intercal = (\Theta_1, \Theta_2, .., \Theta_M) \in \mathbb{R}^M$ and dw_t^Y is the increment of an M-dimensional Wiener process with $\left[dw_t, dw_t^Y\right] = 0$, then, dropping the time index t, the value function of this problem F satisfies the following Hamilton-Jacobi-Bellman equation

$$\begin{aligned}\theta F = \max_{(c,\ \frac{z}{W} \in Q)} \quad & \tfrac{c^{1-\gamma}}{1-\gamma} - cF_1 + (rW + Y)F_1 + mYF_2 + \tfrac{\Sigma^\intercal\Sigma + \Theta^\intercal\Theta}{2} Y^2 F_{22} \\ & + z^\intercal \left((\mu - r\bar{1})F_1 + \sigma\Sigma Y F_{12}\right) + \tfrac{z^\intercal\sigma\sigma^\intercal z}{2} F_{11}.\end{aligned}$$

Observe that this equation is the same as the one derived when labor income is spanned, with the exception of an additional term $\frac{1}{2}\Theta^\intercal\Theta Y^2 F_{22}$. This last term has an impact on the dynamics of the Hamilton-Jacobi-Bellman equation but does not alter the maximization problem. This indicates that asset selection *does* occur in the case of unspanned stochastic income. Furthermore assets are dropped out of the portfolio in the same order as in the spanned labor income case. The $N+1$ regions described in Proposition I.3 are characterized by the same threshold levels $y_{K,K}$ of the lifetime relative risk aversion. However, the wealth over income ratio cutoff points defining these regions are different.

G. Finite Horizon

Our analysis has focused so far on the infinite horizon case. To accommodate the case of life-cycle consumption and investment we now consider a case where the investor receives an income stream Y only over the period $[0, T]$ with $T > 0$. At time T, the investor retires and no longer receives any income. After date T, death occurs after an additional τ years. We assume that the investor does not have a bequest motive. Since we assume that the margin constraint is not binding when there is no income, after time T the margin constraint can be ignored.[11] At time T the value function B is given by

$$B(W_T, \tau) = \frac{A^\gamma \left(1 - e^{-\tau/A}\right)^\gamma}{1-\gamma} W_T^{1-\gamma},$$

where A is defined in Equation (6).

[11]The investor is still subject to margin requirements after retirement, but given the range of parameters we study, the margin requirements do not bind when income is equal to zero.

For time $t \leq T$ the value function F satisfies

$$F(W_t, Y_t, t) = \max_{(c,\, \frac{z}{W} \in Q)} E_t \left[\int_t^T \frac{c_s^{1-\gamma}}{1-\gamma} e^{-\theta(s-t)} ds + B(W_T, \tau) e^{-\theta(T-t)} \right]$$

under the budget constraint

$$dW_s = \left(rW_s - c_s + Y_s + z_s^\intercal(\mu - r\bar{1}) \right) ds + \sigma z_s^\intercal dw_s$$

with $W_t > 0, Y_t > 0$ given. Note that F is still homogeneous of degree $1-\gamma$ and can be written as $F(W_t, Y_t, t) = Y_t^{1-\gamma} f(v_t, t)$, with $v = W/Y$. Over $[0, T]$, the reduced value function f satisfies the following Hamilton-Jacobi-Bellman equation

$$\begin{aligned}
\left(\theta + (\gamma - 1)(m - \gamma \frac{\Sigma^\intercal \Sigma}{2})\right) f(v,t) = & f_2(v,t) + \frac{\gamma \left(f_1(v,t) \right)^{\frac{\gamma-1}{\gamma}}}{1-\gamma} + f_1(v,t) \\
& + (r - m + \gamma \Sigma^\intercal \Sigma) v f_1(v,t) + \frac{\Sigma^\intercal \Sigma}{2} v^2 f_{11}(v,t) \\
& + \max_{\omega \in Q} \left[\omega^\intercal \left(\eta v f_1(v,t) - \sigma \Sigma Y v^2 f_{11}(v,t) \right) + v^2 f_{11}(v,t) \omega^\top \frac{\sigma \sigma^\intercal}{2} \omega \right],
\end{aligned}$$

with $\omega = z/W$ and boundary condition $f(v_T, T) = B(v_T, \tau)$. The analysis conducted in the infinite horizon case still applies, so asset selection still takes place. However, the lifetime relative risk aversion $y_t = -v f_{11}(v,t)/f_1(v,t)$ is now time dependent, which implies that the thresholds in the wealth to income ratio where the investor drops assets from his portfolio change as the investor approaches retirement.

II. Numerical Algorithm and Results

To quantitatively illustrate our theoretical results we consider a discrete-time example of an investor who receives income over his working life, and who retires at a pre-specified age. The investor has access to a set of risky assets that we calibrate to U.S. industry portfolios. To numerically solve this optimal asset allocation and consumption problem we extend the numerical algorithm proposed by Brandt, Goyal, Santa-Clara, and Stroud (2005) to allow for endogenous state variables and margin constraints. The algorithm is designed to solve optimal control problems using a functional approximation of conditional expectations and is particularly suitable for problems with a large number of state and choice variables. The algorithm proceeds in a dynamic programming fashion, solving the optimal consumption and asset allocation problem backward in time. At each time step the value function is approximated using functional interpolation. The optimal allocation and consumption are computed as solutions of the first order conditions for the problem's value function, augmented by the constraints multiplied by Lagrange multipliers. One key difference in the algorithm, compared to the algorithm by Brandt, Goyal, Santa-Clara, and Stroud (2005), is the introduction of an iterative step to solve the first order conditions: rather than relying on approximating the first order conditions over a large region, we focus our approximation in a neighbourhood of a potential solution. Once the solution is computed, we further restict the neighbourhood of approximation and refine the solution, until a desired accuracy is achieved.[12] We outline the steps of the algorithm below and describe it in detail in the online Appendix.

Algorithm

Step 1: Dynamic Programming

a. For each time step, starting at the terminal time and working backward, construct a grid in the state space and compute the value function and optimal consumption and portfolio decisions for each point in the grid.

b. Approximate the value function on the corresponding grid points. This approximation will be used in earlier time steps to compute conditional expectations of the value function.

Step 2: Karush-Kuhn-Tucker Conditions To solve the Bellman equation for each point on the grid perform the following steps

a. Combine the constraints in the portfolio positions and the evolution of the state variables with the value function in a Lagrangian function with Lagrange multipliers.

b. Make a change of variables that allows the consumption optimization problem to be solved independently of the asset allocation optimization problem.[13]

c. Construct the system of first order conditions (KKT conditions) for the consumption and asset allocation optimization problems.

[12] This improvement in the algorithm by Brandt, Goyal, Santa-Clara, and Stroud (2005) was introduced in Yang (2009).
[13] This change of variables was proposed in a similar problem by Carroll (2006).

d. Find the optimal solution of the Karush-Kuhn-Tucker conditions for the asset allocation optimization problem using an iterative process:

   i. Start by choosing a region in the choice space that includes the optimal portfolio. This choice can be informed by knowledge of the optimal portfolio at nearby grid points at the same time step, or for the same grid point at a later date.

   ii. Find an approximately optimal portfolio by solving the system of KKT conditions. To solve the system of KKT conditions, approximate the conditional expectations in the derivatives of the Lagrangian function using cross-test-solution regression: choose a quasi-random set of feasible allocations. Calculate the required conditional expectations — interpolating the value of the value function in the following time step from the values at the grid points, estimated from Step 1 in the algorithm — for each feasible choice, and project each on a set of basis functions of the choice variables. Solve the resulting system of equations.

   iii - Test Region Contraction. Repeat step (ii) using a smaller region in which feasible portfolio choices are drawn, chosen based on the location of the previously computed approximately optimal portfolio.

   iv. Repeat until the portfolio choice converges.

   v. Given the optimal portfolio choice, compute the optimal consumption choice using the appropriate KKT condition.

A. Calibration

To apply the numerical algorithm, we consider the case of an investor that receives income until age 65, at which point he retires. After retirement the investor has an expected lifetime of 20 years, which matches the data in the 2004 Mortality Table for the Social Security Administration for a 65 year old female, see Social Security Administration (2004). For the base case we assume that income grows deterministically at a constant growth rate of 3% per year, in line with the assumptions in Viceira (2001).[14] We also assume that the investor is not able to either borrow, or short any of the assets, corresponding to parameter values $\lambda^+ = 1, \lambda^- = \infty$.

[14] We will later consider comparative statics with stochastic income.

The opportunity set available to the investor includes five risky assets corresponding to the indices of five industries: Consumer, Manufacturing, High Tech, Health, and Other. To calculate the covariance matrix for each industry we constructed real returns for each industy using the inflation data provided in Robert Shiller's website, see Shiller (2003), to deflate the annual returns of the five industry portfolios between 1927 and 2004, provided in Ken French's website, see French (2008). The expected returns for each industry were computed using the methodology proposed by Black and Litterman (1992), by matching the market capitalization weights for each industry in July 2008, provided in Ken French's website, to the relative weights that a CRRA investor who receives no income would allocate to each industry within his equity portfolio. The risk-free interest rate was computed from the data in Robert Shiller's website to match the realized one year real interest rate between 1927 and 2004.

B. Diversification Measures

Calvet, Campbell, and Sodini (2008) present an empirical analysis of diversification of household portfolios in Sweden, and describe several measures that quantify the degree that investors deviate from mean-variance optimal portfolios. We use the same measures in order to determine the potential magnitude of the impact of the financial constraints on diversification. We present the measures below, following the description and notation in Calvet, Campbell, and Sodini (2008).

Denoting by $r_{h,t}, r_{B,t}$ the returns of the risky asset portfolios of the constrained and unconstrained investors, respectively, we have the following variance decomposition

$$r_{h,t} = \alpha_h + \beta_h r_{B,t} + \varepsilon_{h,t},$$

and, if we denote by σ_B, σ_h the standard deviation of the returns of the equity portfolio of the unconstrained and constrained investors respectively, we have

$$\sigma_h^2 = \beta_h^2 \sigma_B^2 + \sigma_{i,h}^2.$$

The interpretation of this decomposition is that the portfolio of the constrained investor has *systematic risk* $|\beta_h|\sigma_B$ and *idiosyncratic risk* $\sigma_{i,h}$. The *idiosyncratic variance share* is given by

$$\frac{\sigma_{i,h}^2}{\sigma_h^2} = \frac{\sigma_{i,h}^2}{\beta_h^2\sigma_B^2 + \sigma_{i,h}^2}.$$

Another measure of portfolio diversification is the Sharpe ratio of the risky portion of the portfolio. We denote the Sharpe ratio of the portfolio of an investor that does not face financial constraints S_B, and the Sharpe ratio of a constrained investor S_h. These ratios are defined by the ratio of the excess return of the respective portfolio to the standard deviation of excess returns

$$S_h = \frac{\mu_h}{\sigma_h},$$

where μ_h, σ_h, are the excess return and standard deviation of excess return for the portfolio of the constrained investor. The *relative Sharpe ratio loss* is defined by

$$RSRL_h = 1 - \frac{S_h}{S_B}.$$

While the relative Sharpe ratio loss is a measure of the diversification loss in the risky asset portion of the portfolio, it does not necessarily reflect the overall efficiency loss in the portfolio. To capture this loss, we define the *return loss* as the average return loss by the investor by choosing a suboptimal portfolio

$$RL_h = w_h(S_B\sigma_h - \mu_h),$$

where w_h is the portion of the portfolio invested in risky assets.

Finally, we define a measure associated with utility losses for the constrained investor, compared to the unconstrained one. It is defined as the increase in the risk-free rate that would make the constrained investor indifferent between being constrained with the higher risk-free rate and being unconstrained. In the case of a risk-averse investor with CRRA preferences with risk aversion coefficient γ, Calvet, Campbell, and Sodini (2008) calculate the utility loss from the relationship

$$UL_h = \frac{S_B^2 - S_h^2}{2\gamma}.$$

C. Base Case

The optimal asset allocations for the base case parameters, listed in Table III, are presented in Figure 2 for investors 30 and 60 years old over a range of wealth to income ratios.

From Figure 2 we notice that as the financial wealth of the investor decreases compared with his income, the investor allocates a larger proportion of his wealth to the risky assets. For a 30 year old investor the margin constraint binds if the investor's financial wealth is smaller than 12.9 times his annual income. While the proportion in which each risky asset is held within the equity portfolio does not change when the margin constraint is not binding, once the constraint binds the investor shifts his portfolio to increase the portfolio's expected return, sacrificing diversification. When the financial wealth reaches a level of 8.2, 6.5, 3.7, and 0.93 times the investor's annual income, the investor drops the Health, Manufacturing, Consumer, and Other industry indices from his portfolio, respectively. For financial wealth levels below 93% of the investor's annual income, the investor's equity portfolio consists only of the index of the High Tech industry. A similar pattern is observed for an investor of age 60. In that case, since the remaining income spans a smaller number of years; i.e., the discounted value of future earnings is smaller than the 30 year old investor, the constraint binds at a lower level of the financial wealth equal to 2.4 times annual income. For lower levels of the financial wealth to income ratio the 60 year old investor also shifts his equity portfolio, dropping the Health, Manufacturing, Consumer, and Other industry indices at ratios of 1.8, 1.6, 1.1, and 0.4 respectively.

Table III presents further details of the optimal allocations for different levels of the financial wealth to annual income ratio, as well as values for the various diversification measures and the investor's life-time relative risk aversion. From the table we notice that when the margin constraint is not binding and the ratio of financial wealth to income decreases, the investor increases the portfolio's expected return by increasing the percentage of his wealth invested in risky assets while maintaining a diversified portfolio. Once the constraint binds, further reductions in the financial wealth to annual income ratio result in a deterioration of the portfolio diversification measures. As an example, a 30 year old investor whose financial wealth is equal to one year of his labor income holds a portfolio that has 11.1% idiosyncratic volatility — which corresponds to 11.3% of the portfolio's variance — Sharpe ratio of 25.9% compared to 27.3% achieved when the portfolio is diversified, and a return loss of 48 basis points per year. We point out that, even though the volatility of the equity portfolio of the constrained investor has higher volatility, its size is smaller than the equity portfolio of the unconstrained investor. Due to the smaller size of the equity portfolio, shocks to the prices of the risky assets have a smaller effect to the wealth of the constrained investor, leading to a smoother consumption choice. The beta of the investor's equity portfolio is 14% greater than the beta of the equity part of the diversified portfolio, while the lifetime relative risk aversion of the investor is 0.23, close to that of a risk-neutral investor.

Panel B of Table III presents allocations and diversification measures for a 60 year old investor. The results are qualitatively similar to the results in Panel A, with the main difference being that the margin constraint binds at lower levels of the financial wealth to annual income ratio.

Table III presents results obtained by simulating the evolution of the portfolio of an investor starting at age 20. From Panel A we notice that the investor whose financial wealth at age 20 was twice his annual income holds, at age 30, a portfolio that almost always consists of one or two risky assets. At the same time the investor consumes slightly more than his annual labor income. At age 45 the investor starts consuming less than his annual income and saves the remainder. His portfolio is still mostly constrained by the margin requirements. At age 60 the investor has accelerated his saving behavior and is mostly unconstrained in his financial portfolio.[15]

Panel B of Table III presents the simulation results for an investor whose financial wealth at age 20 is equal to ten times his annual income. Even though this investor is relatively richer than the investor in Panel A, the margin constraint still largely binds at age 30, leading to the investor holding an under-diversified equity portfolio. Given his large financial wealth, this investor postpones saving much longer than the investor in Panel A. Overall, the results in both panels indicate that younger investors, even if they have significant amounts of financial wealth, are holding portfolios far from those held by older, unconstrained, investors.

[15]Since consumption is measured with respect to current annual labor income and since in this example income increases 3% annually, the reduction in consumption relative to income observed in the table does not necessarily imply a reduction in the actual amount consumed by the investor. Nevertheless, consumption to income ratios above 1.0 imply that the investor consumes part of his financial wealth while ratios below 1.0 imply that the investor saves part of his labor income.

D. Comparative Statics

Regulation T Margin Constraints

Figure 3 presents the optimal asset allocations for an investor facing a margin constraint in line with the requirements in Regulation T of 50% for long positions and 150% for short positions. For the calibrated parameter values from Table III, the investor never shorts any of the risky assets. Compared to Figure 2, the investor is unconstrained for a greater range of his financial wealth to income ratio, with the allocations being identical when the margin constraint does not bind for either investor. The margin constraint for the investor that faces the Regulation T margin requirements binds at a level of financial wealth equal to 2.3 times his annual income at age 30 and 91% of his annual income at age 60. The order that assets drop out is the same as in the base case, and the last asset held in the portfolio is the High Tech industry index, which is exclusively held at levels of financial wealth below 18% of annual income at age 30 and below 12% of annual income at age 60.

Stochastic Income

Figure 4 presents the optimal asset allocations when the annual standard deviation of income growth is 10%, and income growth is uncorrelated with the returns of the risky assets, in line with the case studied in Viceira (2001). The remaining parameters are the same as in the base case, given in Table III. From the figure we notice that stochastic income has an effect in asset allocations: both for age 30 and age 60 investors, allocations in the industry indices are reduced compared to the base case of deterministic income, an effect intuitively expected due to the higher risk implied by the stochastic nature of income growth. While, in line with our theoretical results, the order in which assets are dropped from the equity portfolio when the ratio of financial wealth to income decreases is the same as in Figure 2, the threshold when the margin constraint binds is lower. For a 30 year old investor who receives income with deterministic growth the margin binds at a financial wealth level equal to 12.9 times his annual income, while for the investor who receives income with stochastic growth the margin constraint binds at a level of financial wealth equal to 10.4 times his annual income. An intuitive explanation for this reduction is that since income growth is uncorrelated with asset returns, income helps in diversifying the investor's portfolio, implying that the margin requirement is less onerous.

Overall, Figure 4 illustrates that even in the case of stochastic income the intuition developed by our theoretical results in Section I remains valid; i.e., an investor with low levels of financial wealth compared to labor income holds under-diversified portfolios consisting of only a few out of many possible risky assets.

Non-negative Wealth

A case of constrained choice previously studied in the literature is the case when the investor's wealth is required to remain greater or equal to zero but where the investor does not face a margin requirement, see He and Pagès (1993), El Karoui and Jeanblanc-Picqué (1998) and Duffie, Fleming, Soner, and Zariphopoulou (1997). The margin requirement is a stricter constraint, since it automatically guarantees non-negative wealth. To quantify the difference in asset allocations, Figure 5 presents the optimal asset allocation for an investor facing a non-negative wealth constraint, but who is otherwise identical to our basecase investor. From the figure we notice that in both the cases of a non-negative wealth constraint and of a margin requirement, investment in risky assets increases as the wealth to income ratio decreases. On the other hand, there are significant differences: unlike the case of a margin requirement, an investor that faces a non-negative wealth constraint maintains a diversified portfolio, even when his income is much greater than his wealth; in addition, the size of the risky asset portfolio for the investor that faces a non-negative wealth constraint is much larger than for an investor that faces a margin requirement. In order to finance this larger investment in risky assets, the results in the figure indicate that the investor that is constrained to maintain wealth non-negative borrows amounts up to 10 times his wealth or more, using his income as collateral.

III. Conclusion

The results we have presented indicate that financial constraints can be a significant determinant of individual portfolios, and can, to some extent, account for empirical findings. The variable that is instrumental in the determination of the portfolios, and the extent to which investors deviate from diversified portfolios, is the ratio of current wealth to income. For large values of this ratio the investor is unconstrained, while the constraint has the largest effect at low values of the ratio. This result implies that young investors are most likely to be affected while older investors are more likely to hold diversified portfolios. This prediction is in line with several empirical papers. For example, Goetzmann and Kumar (2008) and Calvet, Campbell, and Sodini (2008) report that age is a significant determinant of under-diversification. Kumar (2009) reports that young investors are more likely to hold stocks with lottery-like payoffs that seemingly expose them to uncompensated risk. Goetzmann and Kumar (2008) report that households that only have a retirement investment account, which presumably includes households that do not have enough wealth for an additional investment account, hold more under-diversified portfolios. Our findings also provide a rational explanation for the empirical finding that investors only hold a small number of stocks in their portfolio: similar to Black (1972), constrained investors try to increase their expected return at the cost of holding less diversified portfolios by shifting toward portfolios with higher β.[16] Ivković, Sialm, and Weisbenner (2008) show that, while investors hold relatively few stocks in their portfolios, the number rises with an increase in account balance, which can be thought of as a proxy of current wealth.

Beyond the existing empirical literature, our theoretical and numerical results also provide several empirically testable predictions. For example, the calibration predicts that severely constrained investors; i.e., those with a very low wealth to income ratio, will hold only the asset with the highest expected return, which is not the one with the highest Sharpe ratio. As we already mentioned, Ivković, Sialm, and Weisbenner (2008) report that concentrated portfolios have lower Sharpe ratios. It would be interesting to also determine whether they also have higher expected returns. Another example would be to test whether investors that borrow on margin — a possible indication that the investor is financially constrained — hold less diversified portfolios than investors that do not.

An additional empirical prediction involves the dynamics of underdiversification: given that our model predicts that the degree of underdiversification of an investor's portfolio depends on the ratio of his labor income to his financial wealth, we would expect underdiversification to decrease following a negative shock to an investor's labor income, such as the loss of a job.

While our findings reveal a clear link between the combination of labor income and financial constraints and under-diversified portfolios, several hurdles remain before a rational model can explain all

[16] An interesting question is whether the inclusion of put options, with their higher leverage, would alleviate the financial constraint. While options have not empirically been a significant component of individual portfolios, the reason why investors shun them is unclear, and can be, for example, due to their high prices, see Kubler and Willen (2006). This question is outside the scope of our paper.

the available empirical findings. One challenge is the conflict between the theoretical prediction that investors will not hold the riskless asset and an undiversified equity portfolio simultaneously, and the empirical results reported in Polkovnichenko (2005) and Calvet, Campbell, and Sodini (2008). While our model cannot address this issue, a possible resolution could be a model with an additional cost imposed on trading risky assets. Such a cost could be due, for example, to transaction costs, or capital gain taxes.[17] An alternative explanation would be the combination of a desired minimum level of wealth and financial constraints, see Liu (2009).[18]

Another issue that is not addressed directly in our paper is the holding of individual stocks in investor portfolios rather than mutual funds or exchange-traded funds (ETFs). Beyond forced holdings, such as company stock granted to employees, individual stock selection could be predicted in a rational framework by including competition for scarce local resources, creating a home-bias effect in areas where local companies grant stocks to their employees they cannot trade out of — see DeMarzo, Kaniel, and Kremer (2004).[19]

In addition to rational explanations, it is likely that behavioral based explanations have a significant effect. Our contribution in this paper is to offer the ratio of wealth over income as a variable that can be used to understand underdiversification in investor portfolios. It would be important to find additional variables that can distinguish between the rational and behavioral explanations.

An interesting extension of our work would be to consider assets with different margin requirements. In this case, we expect that the assets that have the highest return, when leveraged to the greatest extent possible, would appear most attractive to constrained investors. Such behavior would be in line with the preference of individual investors for residential real estate investments over financial investments, due to the lower margin requirements for residential real estate.

[17] In Gallmeyer, Kaniel, and Tompaidis (2006) it is shown that capital gain taxation can also induce an investor to hold an underdiversified portfolio, while simultaneously holding the riskless asset.

[18] The difference between the prediction of underdiversification in our paper and the prediction of underdiversification in Liu (2009) is that, in our work, the ratio of financial wealth to income determines the degree of underdiversification, while in Liu (2009) it is the desire for precautionary savings, driven by the difference between wealth and the desired minimum level of consumption. The implication, for empirical studies, is that both wealth and wealth over income are potential explanatory variables for observed underdiversification.

[19] Nieuwerburgh and Veldkamp (2010) propose another possible explanation. Assuming that investors have limited resources to learn about individual stocks, they show that it is optimal to focus on a small subset of stocks, and hold portfolios that simultaneously include a diversified fund and a concentrated set of assets.

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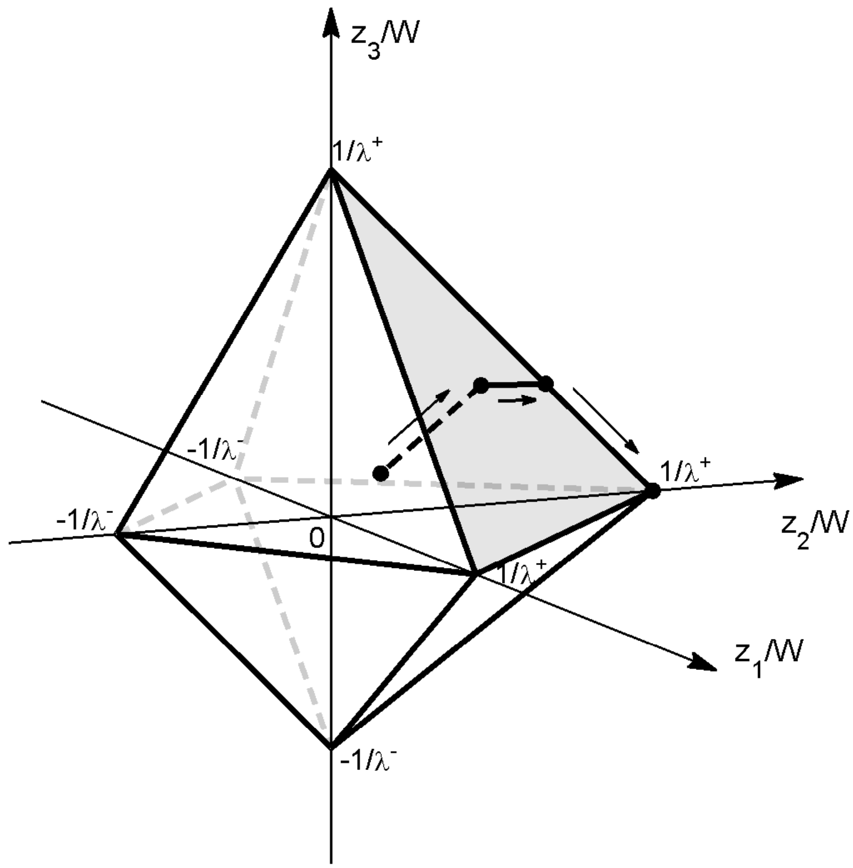


Figure 1. This figure illustrates the intuition of Proposition I.3 for the case of an asset allocation problem with a margin requirement, deterministic income and three risky assets. The axes correspond to the allocations in each risky asset as a percentage of wealth $z_i/W, i = 1,2,3$. The margin coefficients are λ^+ for long positions and λ^- for short positions. The allocations are shown at different values of the wealth to income ratio, with the arrows indicating the direction of change in the allocations as the ratio decreases. The margin requirement binds when the chosen allocation lies on the shaded plane; asset 1 is dropped when the allocation lies on the edge on the $z_2/W - z_3/W$ plane. Asset 3 is dropped when income is much larger that wealth, and the allocation is represented by the vertex on the positive z_2/W axis.

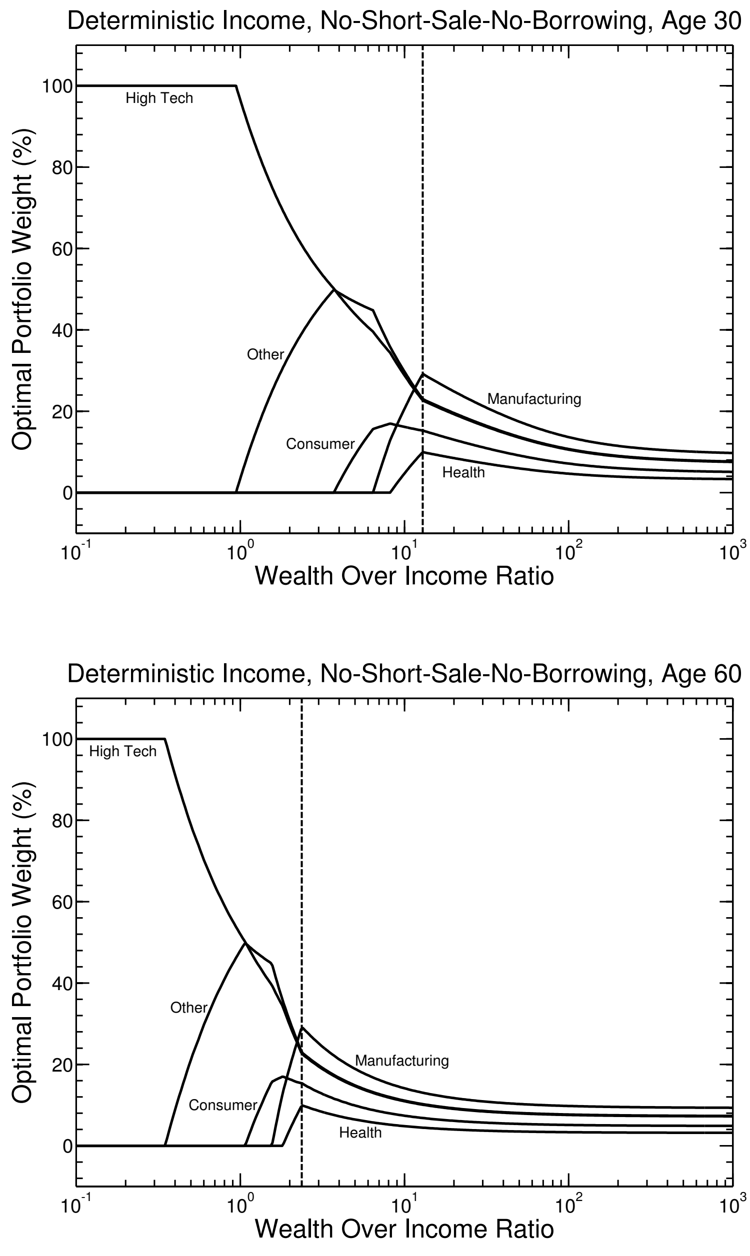


Figure 2. This figure presents the asset allocations for different levels of the financial wealth to annual income ratio. The investor receives deterministic income. The investor's opportunity set consists of a riskless asset and five risky assets calibrated to the returns of stock industry indices for the industries High Tech, Consumer, Manufacturing, Health, and Other. The parameter values for the processes followed by the risky and riskless assets are given in Table III. The investor is not allowed to borrow or short an asset and is required to pay 100% of an asset's value. The top panel corresponds to a 30 year old investor and the bottom panel to a 60 year old investor.

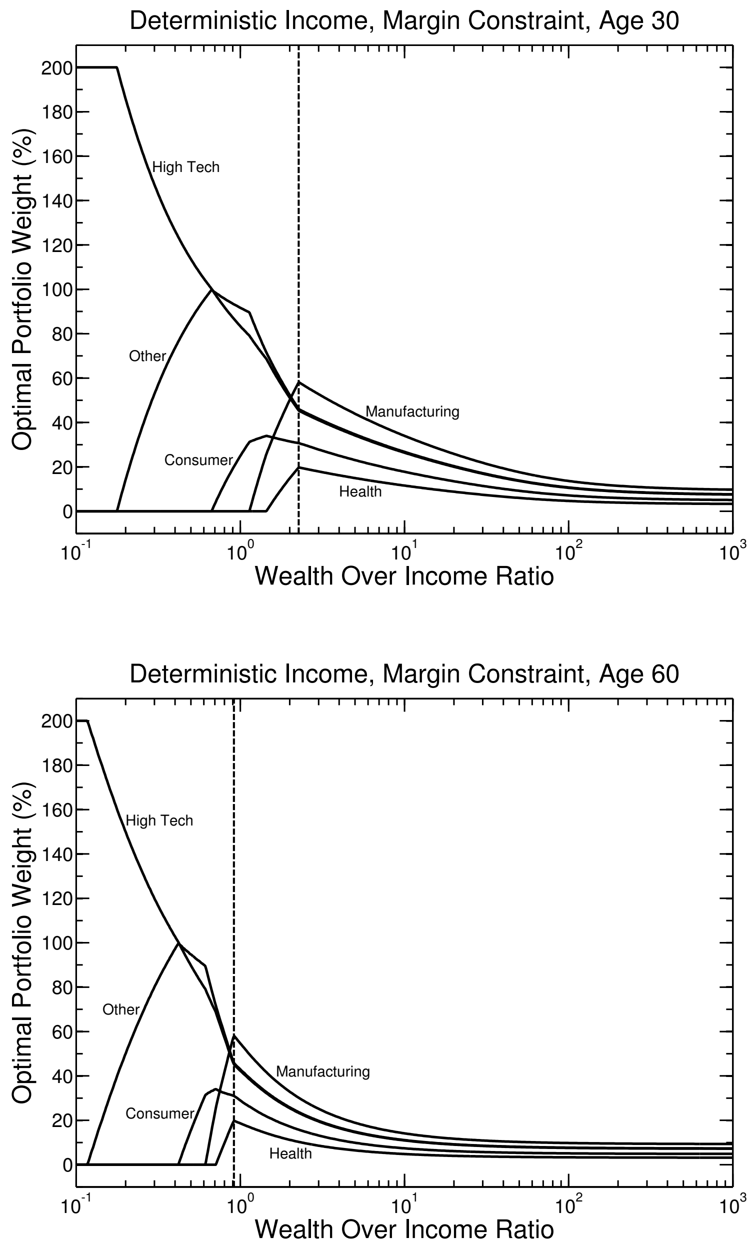
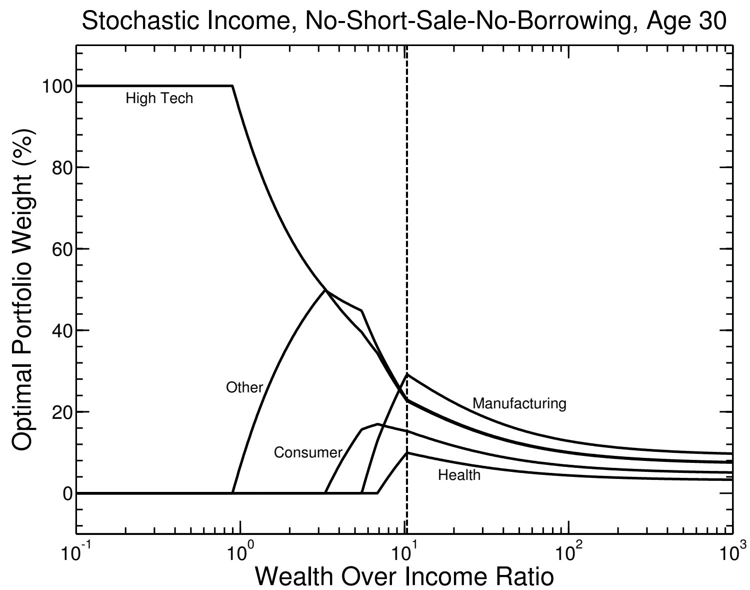


Figure 3. This figure presents the asset allocations for different levels of the financial wealth to annual income ratio. The investor receives deterministic income. The investor's opportunity set consists of a riskless asset and five risky assets calibrated to the returns of stock industry indices for the industries High Tech, Consumer, Manufacturing, Health, and Other. The parameter values for the processes followed by the risky and riskless assets are given in Table III. The investor is allowed to purchase a risky asset on 50% margin. The top panel corresponds to a 30 year old investor and the bottom panel to a 60 year old investor.



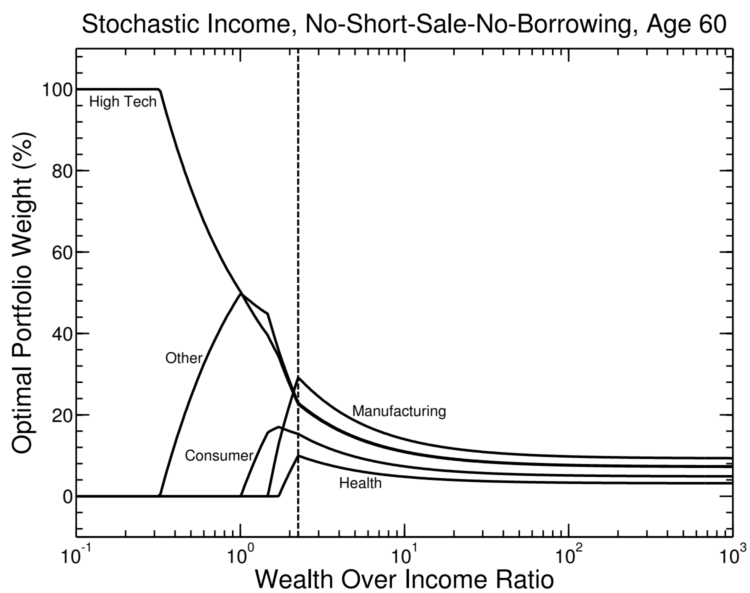


Figure 4. This figure presents the asset allocations for different levels of the financial wealth to annual income ratio. The investor receives stochastic income with annualized volatility 10%. The investor's opportunity set consists of a riskless asset and five risky assets calibrated to the returns of stock industry indices for the industries High Tech, Consumer, Manufacturing, Health, and Other. The parameter values for the processes followed by the risky and riskless assets are given in Table III. The investor is not allowed to borrow or short an asset and is required to pay 100% of an asset's value. The top panel corresponds to a 30 year old investor and the bottom panel to a 60 year old investor.

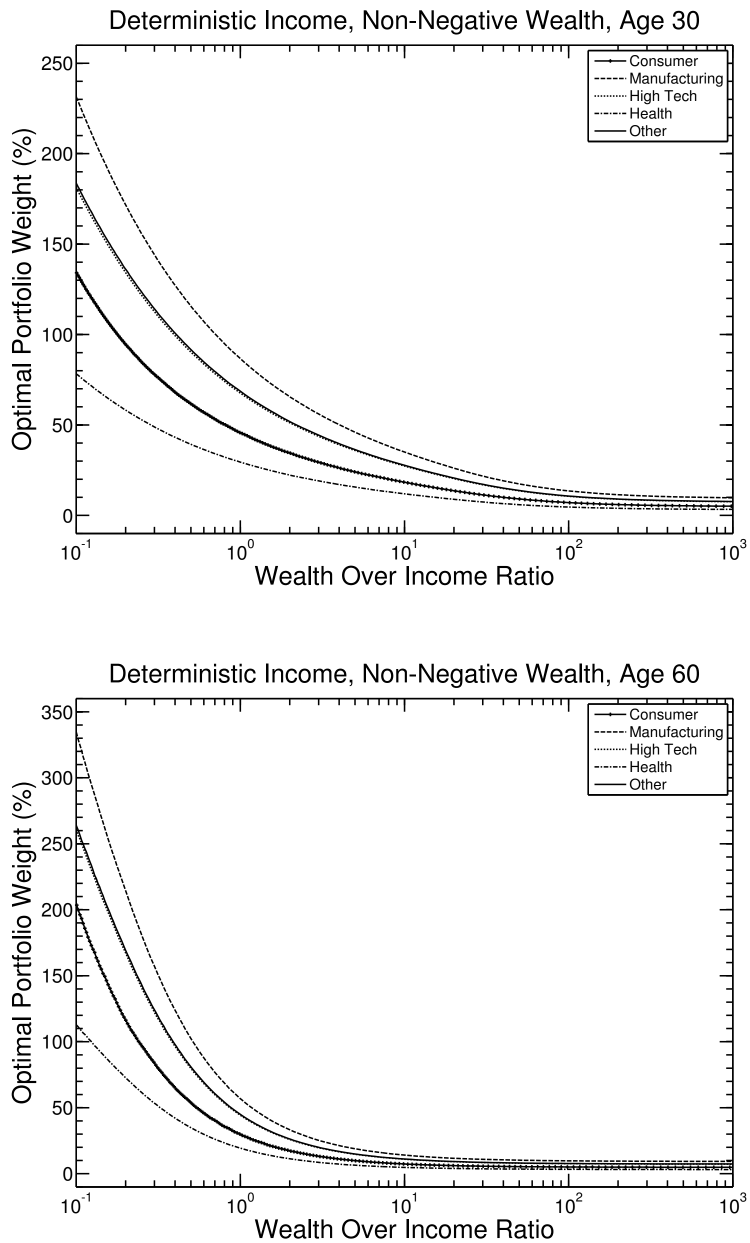


Figure 5. This figure presents the asset allocations for different levels of the financial wealth to annual income ratio for an investor that faces a non-negative wealth constraint but no borrowing or margin constraints. The investor receives deterministic income and has access to a riskless asset and five risky assets calibrated to the returns of stock industry indices for the industries High Tech, Consumer, Manufacturing, Health, and Other. The parameter values for the processes followed by income and the risky and riskless assets are given in Table III. The top panel corresponds to a 30 year old investor and the bottom panel to a 60 year old investor.

Table 1
Parameter Values for the Base Case

Number of periods	45 years (age 20 to age 65)				
Risk aversion					3
Long margin					1
Short margin					∞
Time discount factor (annually)					0.98
Interest rate (annually)					1.4%
Income growth rate (annually)					3%
	Cnsmr	Manuf	HiTec	Hlth	Other
Asset drift (annually)	8.51%	7.83%	9.51%	6.97%	8.87%
Asset volatility (annually)	28.9%	25.8%	33.3%	26.6%	29.7%
Correlations between assets	1.000	0.898	0.832	0.732	0.932
		1.000	0.848	0.698	0.930
			1.000	0.772	0.856
				1.000	0.727
					1.000

Table 2: Asset Allocation and Diversification Measures of the Base Case

This table presents the optimal asset allocations and diversification measures for the base case : deterministic income with no-short-sale-no-borrowing constraint. W/Y is the current wealth to income ratio. Cnsmr, Manuf, HiTec, Hlth, and Other are the portfolio weights (as a percentage of current wealth) of the five industry indices: Consumer, Manufacturing, High Tech, Health, and Other. Margin is the total usage of the margin account in percentage. μ_h and σ_h are the expected value and standard deviation of the excess return of the risky part of the portfolio. $\sigma_{i,h}$ is the idiosyncratic standard deviation. IVarS is the idiosyncratic variance share. S_h is the Sharpe ratio of the risky part of the portfolio. RSRL_h is the relative Sharpe ratio loss. RL_h is the return loss of the total portfolio. UL_h is the utility loss. β_h is the β of the constrained portfolio with respect to the unconstrained portfolio. LRRA is the lifetime relative risk aversion.

Panel A: Age 30

W/Y	Cnsmr (%)	Manuf (%)	HiTec (%)	Hlth (%)	Other (%)	Margin (%)	μ_h (%)	σ_h (%)	$\sigma_{i,h}$ (%)	IVarS (%)	S_h (%)	RSRL_h (%)	RL_h (%)	UL_h (%)	β_h	LRRA
∞	5	9	7	3	7	32	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	3.00
1000.0	5	10	8	3	8	33	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	2.87
100.0	7	14	11	5	11	47	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	2.05
20.0	13	25	19	8	20	85	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	1.13
15.0	14	28	21	9	22	95	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	1.02
12.0	15	27	24	9	25	100	7.49	27.4	0.3	0.0	27.3	0.0	0.00	0.00	1.01	0.92
10.0	16	21	29	5	30	100	7.67	28.1	1.3	0.2	27.3	0.0	0.00	0.00	1.03	0.87
9.0	17	17	32	2	33	100	7.78	28.5	1.8	0.4	27.3	0.1	0.00	0.00	1.05	0.82
8.0	17	12	35	0	37	100	7.91	29.0	2.5	0.7	27.3	0.2	0.01	0.00	1.06	0.80
7.0	16	5	38	0	42	100	8.01	29.4	3.1	1.1	27.2	0.3	0.02	0.01	1.08	0.77
6.0	14	0	41	0	45	100	8.10	29.8	3.6	1.5	27.2	0.5	0.04	0.01	1.09	0.73
5.0	9	0	44	0	47	100	8.14	29.9	3.9	1.7	27.2	0.6	0.05	0.01	1.09	0.63
4.0	2	0	48	0	49	100	8.20	30.2	4.6	2.3	27.1	0.8	0.07	0.02	1.10	0.56
3.0	0	0	55	0	45	100	8.25	30.5	5.2	2.9	27.0	1.1	0.09	0.03	1.11	0.49
2.5	0	0	59	0	41	100	8.28	30.7	5.7	3.5	26.9	1.4	0.12	0.03	1.11	0.44
2.0	0	0	66	0	34	100	8.33	31.1	6.6	4.5	26.8	1.9	0.16	0.05	1.12	0.41
1.7	0	0	72	0	28	100	8.37	31.4	7.4	5.5	26.7	2.4	0.21	0.06	1.12	0.37
1.4	0	0	80	0	20	100	8.42	31.9	8.5	7.1	26.4	3.2	0.28	0.08	1.13	0.29
1.2	0	0	87	0	13	100	8.47	32.3	9.6	8.8	26.2	4.1	0.36	0.10	1.14	0.28
1.0	0	0	96	0	4	100	8.54	33.0	11.1	11.3	25.9	5.3	0.48	0.13	1.14	0.23
0.7	0	0	100	0	0	100	8.57	33.3	11.6	12.2	25.7	5.8	0.53	0.14	1.15	0.19
0.4	0	0	100	0	0	100	8.57	33.3	11.6	12.2	25.7	5.8	0.53	0.14	1.15	0.09
0.0	0	0	100	0	0	100	8.57	33.3	11.6	12.2	25.7	5.8	0.53	0.14	1.15	0.00

Panel B: Age 60																
W/Y	Cnsmr (%)	Manuf (%)	HiTec (%)	Hlth (%)	Other (%)	Margin (%)	μ_h (%)	σ_h (%)	$\sigma_{i,h}$ (%)	IVarS (%)	S_h (%)	RSRL$_h$ (%)	RL$_h$ (%)	UL$_h$ (%)	β_h	LRRA
∞	5	9	7	3	7	32	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	3.00
1000.0	5	9	7	3	7	32	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	2.98
100.0	5	10	8	3	8	33	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	2.85
20.0	6	12	9	4	9	40	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	2.35
15.0	7	13	10	4	10	43	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	2.22
12.0	7	13	10	5	10	46	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	2.07
10.0	7	14	11	5	11	48	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	1.97
9.0	8	15	11	5	12	50	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	1.87
8.0	8	15	12	5	12	52	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	1.81
7.0	8	16	13	6	13	55	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	1.71
6.0	9	17	13	6	14	59	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	1.62
5.0	10	19	15	6	15	65	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	1.48
4.0	11	21	16	7	17	73	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	1.29
3.0	13	25	20	9	20	86	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	1.08
2.5	15	28	22	10	22	97	7.43	27.2	0.0	0.0	27.3	0.0	0.00	0.00	1.00	0.98
2.0	16	19	30	4	31	100	7.71	28.2	1.5	0.3	27.3	0.0	0.00	0.00	1.04	0.83
1.7	16	8	36	0	39	100	7.97	29.2	2.8	0.9	27.3	0.2	0.02	0.01	1.07	0.74
1.4	12	0	42	0	46	100	8.12	29.8	3.8	1.6	27.2	0.5	0.04	0.01	1.09	0.67
1.2	5	0	47	0	48	100	8.17	30.1	4.3	2.0	27.1	0.7	0.06	0.02	1.10	0.57
1.0	0	0	52	0	48	100	8.23	30.4	5.0	2.7	27.0	1.0	0.08	0.02	1.10	0.56
0.7	0	0	64	0	36	100	8.31	31.0	6.3	4.2	26.9	1.7	0.15	0.04	1.11	0.44
0.4	0	0	91	0	9	100	8.51	32.6	10.3	9.9	26.1	4.6	0.41	0.11	1.14	0.22
0.0	0	0	100	0	0	100	8.57	33.3	11.6	12.2	25.7	5.8	0.53	0.14	1.15	0.00

Table 3
Simulation Results of Base Case

This table presents summary statistics of the simulated wealth as well as the portfolio and consumption choices for an individual investor starting from a given initial wealth and following the optimal investment and consumption strategy. The results are based on 10,000 simulation paths. W/Y and C/Y are the realized wealth to income ratio and consumption to income ratio. Cnsmr, Manuf, HiTec, Hlth, and Other are the portfolio weights, as a percentage of current wealth, of the five industry indices: Consumer, Manufacturing, High Tech, Health, and Other. Margin is the total usage of the margin account in percentage. Q_{25}, Q_{50}, and Q_{75} are the 25% percentile, the 50% percentile (median), and the 75% percentile. SD is the standard deviation.

		W/Y	C/Y	Cnsmr (%)	Manuf (%)	HiTec (%)	Hlth (%)	Other (%)	Margin (%)
Panel A: Initial wealth equal to two years of income									
Age 30	Q_{25}	0.3	1.0	0	0	78	0	0	100
	Q_{50}	0.7	1.1	0	0	100	0	0	100
	Q_{75}	1.4	1.1	0	0	100	0	22	100
	Mean	1.2	1.1	1	0	88	0	11	100
	SD	1.7	0.1	3	2	19	1	16	1
Age 45	Q_{25}	1.0	0.8	0	0	59	0	0	100
	Q_{50}	1.5	0.9	0	0	78	0	20	100
	Q_{75}	2.5	0.9	0	0	100	0	37	100
	Mean	2.3	0.9	2	1	76	0	21	100
	SD	2.9	0.1	4	5	22	2	17	4
Age 60	Q_{25}	5.2	0.6	7	14	11	5	11	49
	Q_{50}	7.1	0.7	8	16	12	5	13	55
	Q_{75}	9.9	0.8	10	19	14	6	15	64
	Mean	8.1	0.7	9	17	13	6	13	58
	SD	4.4	0.2	2	4	3	1	4	13
Panel B: Initial wealth equal to ten years of income									
		W/Y	C/Y	Cnsmr (%)	Manuf (%)	HiTec (%)	Hlth (%)	Other (%)	Margin (%)
Age 30	Q_{25}	2.7	1.2	0	0	26	0	21	100
	Q_{50}	5.7	1.4	11	0	42	0	33	100
	Q_{75}	11.3	1.7	15	21	57	7	45	100
	Mean	8.4	1.5	8	9	45	3	32	97
	SD	8.2	0.4	7	11	23	4	14	8
Age 45	Q_{25}	1.9	0.9	0	0	22	0	15	95
	Q_{50}	4.0	1.0	6	0	46	0	26	100
	Q_{75}	10.6	1.3	12	18	69	6	42	100
	Mean	8.4	1.2	6	8	49	2	27	93
	SD	10.5	0.4	6	10	28	3	15	15
Age 60	Q_{25}	6.3	0.6	7	13	10	4	10	43
	Q_{50}	9.4	0.8	8	14	11	5	11	50
	Q_{75}	14.7	1.1	9	17	13	6	13	58
	Mean	11.9	0.9	8	15	12	5	12	52
	SD	8.9	0.5	2	4	3	1	3	13