Asset Selection and Under-Diversification with Financial Constraints and Income: Implications for Household Portfolio Studies

Online Appendix

A. Proof of Proposition I.1

Assume that $(c,(x,z))$ is a feasible strategy for initial conditions (W_t,Y_t). Then, for all $\alpha > 0$, we show that $(\alpha c,(\alpha x,\alpha z))$ is a feasible strategy for initial conditions $(\alpha W_t,\alpha Y_t)$. Consider the dynamics for the wealth process W_α, with initial conditions $(\alpha W_t,\alpha Y_t)$, following the consumption and investment plan $(\alpha c,(\alpha x,\alpha z))$. We have

$$dW_{\alpha s} = \alpha W_s ds - \alpha c_s ds + \alpha Y_s ds + \alpha z_s^{\intercal}(\mu - r\overline{1})ds + \alpha z_s^{\intercal}\sigma dw_s = \alpha dW_s,$$

therefore, $W_{\alpha s} = \alpha W_s$. Similarly, we have $Y_{\alpha s} = \alpha Y_s$. The investment strategy satisfies the margin requirement since $\lambda^{\intercal}(\alpha z) = \alpha\lambda^{\intercal} z \leq \alpha W$. It follows that

$$F(\alpha W,\alpha Y) \leq \alpha^{1-\gamma}F(W,Y), \tag{1}$$

since the utility function in homogenous of degree $1-\gamma$. In addition

$$F(W,Y) = F(\alpha^{-1}\alpha W,\alpha^{-1}\alpha Y) \leq \alpha^{\gamma-1}F(\alpha W,\alpha Y),$$

so given (1) in fact we have $F(\alpha W,\alpha Y) = \alpha^{1-\gamma}F(W,Y)$.

B. Proof of Proposition I.2

To show that F is non-decreasing in (W_t,Y_t) is simple, since given an initial endowment (W_t,Y_t), it is easy to see that starting with wealth $W_t' > W_t$ or income $Y_t' > Y_t$ at time t, the optimal strategy for the initial condition (W_t,Y_t) is still admissible and potentially non-optimal for the problem with initial conditions (W_t',Y_t'). This implies that F is non-decreasing in W and Y. To show concavity, consider two initial conditions (W_t,Y_t) and (W_t',Y_t') and $\alpha \in (0,1)$. Denote $(c,(x,z))$ and $(c',(x',z'))$ the optimal strategies respectively for the two initial conditions. Then, the strategy $S : (\alpha c + (1-\alpha)c', \alpha x + (1-\alpha)x', \alpha z + (1-\alpha)z')$ is admissible for the initial condition $I : (\alpha W_t + (1-\alpha)W_t', \alpha Y_t + (1-\alpha)Y_t')$. Denoting W^{α} the wealth process associated with strategy S and initial condition I, for all times s, we have $W_s^{\alpha} = \alpha W_s + (1-\alpha)W_s'$ and similarly for the income process $Y_s^{\alpha} = \alpha Y_s + (1-\alpha)Y_s'$. The margin constraint is satisfied since

$$\lambda^{\intercal}(\alpha z + (1-\alpha)z') = \alpha\lambda^{\intercal}z + (1-\alpha)\lambda^{\intercal}z' \leq \alpha W + (1-\alpha)W \leq W,$$

as both z and z' are feasible. Finally, by strict concavity of the utility function u, we have

$$E_t\left[\int_t^{\infty} u\left(\alpha c_s + (1-\alpha)c_s'\right) e^{-\theta s} ds\right] > E_t\left[\int_t^{\infty} \left(\alpha u(c_s) + (1-\alpha)u(c_s')\right) e^{-\theta s} ds\right],$$

which implies that

$$F(\alpha W_t + (1-\alpha)W_t', \alpha Y_t + (1-\alpha)Y_t') > \alpha F(W_t, Y_t) + (1-\alpha)F(W_t', Y_t').$$

C. Proof of Proposition I.3

As mentioned in the text, the margin constraint is equivalent to 2^N linear constraints of the form

$$\lambda^{\intercal}\frac{z}{W} \leq 1,$$

where $\lambda^{\intercal} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ with $\lambda_k \in \{\lambda^+, -\lambda^-\}$ for $k = 1, 2, \ldots, N$. Each linear constraint is defined by its vector λ. Note that *at most* N constraints can be binding at the same time. If exactly 2 constraints, constraints p and q respectively defined by vectors $\lambda^{(p)}$ and $\lambda^{(q)}$, are binding, it must be the case that vectors $\lambda^{(p)}$ and $\lambda^{(q)}$ have $N-1$ components in common; if the kth component $\lambda_k^p \neq \lambda_k^q$, then $z_k^* = 0$, i.e. asset k is dropped out of the portfolio. More generally, if exactly $K+1$ constraints are binding, K assets have been dropped out of the portfolio; i.e., their allocation is zero, and the vectors $\left\{\lambda^{(i)}\right\}_{i=1}^{K+1}$ of the binding constraints must have $N-K$ components in common. One important implication is that if one asset is optimally dropped out of the portfolio, it is never optimal to hold this asset again when more constraints are binding. The Hamilton-Jacobi-Bellman equation for the primal value function F is

$$\theta F = \max_{\frac{z}{W}\in Q} \quad \frac{\gamma (F_1)^{\frac{\gamma-1}{\gamma}}}{1-\gamma} + (rW+Y)F_1 + mYF_2 + \frac{\Sigma^\intercal\Sigma}{2}Y^2F_{22}$$
$$+z^\intercal\left((\mu - r\overline{1})F_1 + \sigma\Sigma Y F_{12}\right) + \frac{z^\intercal\sigma\sigma^\intercal z}{2}F_{11}$$

Recall that $F(W,Y) = Y^{1-\gamma}f(\frac{W}{Y})$, and $y = -\frac{WF_{11}}{F_1} = -\frac{vf''(v)}{f'(v)}$. The maximization program is equivalent to

$$\max_{\omega\in Q} \quad \omega^\intercal(\eta + y\sigma\Sigma) - \tfrac{y}{2}\omega^\intercal\sigma\sigma^\intercal\omega, \tag{2}$$

with $\omega = z/W$. Program (2) is well defined, since, for $y > 0$, the objective function is strictly concave and the margin constraint is convex, so there is a unique solution. Observe that for $y > 0$ small enough, assuming $\eta \neq 0$, the obvious optimal solution is $\omega^* = (0,\ldots,\omega_k^*,\ldots,0)$, with $\omega_k^* = 1/\lambda_k$, where asset k is such that $\eta_k/\lambda_k = \max_{i=1,..,N}(\eta_i/\lambda_i)$. For y small enough, only one asset is held in the portfolio.

Case $\eta = 0$.

In this case, program (2) is independent of the parameter y, so the fraction of wealth invested in each asset is constant. The unconstrained allocation is $z/W = (\sigma\sigma^\intercal)^{-1}\sigma\Sigma$. If $\max_{\lambda\in\Lambda}\left(\lambda^\intercal(\sigma\sigma^\intercal)^{-1}\sigma\Sigma\right) \leq 1$, the margin constraint is never binding, so $z^*/W = (\sigma\sigma^\intercal)^{-1}\sigma\Sigma$. If, on the other hand, $\max_{\lambda\in\Lambda}\left(\lambda^\intercal(\sigma\sigma^\intercal)^{-1}\sigma\Sigma\right) >$ 1, the constraint is always binding. Depending of the parameters values, K assets are optimally held in the portfolio, with $K = 1,\ldots,N$. More specifically, assuming that asset N is first dropped out, followed by asset $N-1$ and so on, K assets remain in the portfolio if and only if for exactly K assets

$$\max_{\lambda\in\Lambda}\left(\lambda_k e_k^\intercal I_K \frac{z^*}{W}\right) > 0, \quad k = 1,\ldots,K$$

with

$$I_K\frac{z^*}{W} = \frac{(I_K\sigma\sigma^\intercal I_K^\intercal)^{-1}I_K\sigma\Sigma + (1-\lambda^\intercal I_K^\intercal(I_K\sigma\sigma^\intercal I_K^\intercal)^{-1}I_K\sigma\Sigma)I_K\lambda}{\lambda^\intercal I_K^\intercal(I_K\sigma\sigma^\intercal I_K^\intercal)^{-1}I_K\lambda}$$

and $z_k^* = 0$ for $k = K+1,\ldots,N$. The proof is the same as in the case $\eta \neq 0$ and is therefore omitted.

Case $\eta \neq 0$.

Since we intend to achieve a maximum, the smaller the number of constraints that are binding, the higher the maximum value. First we look at the values of y such that the margin constraint is not binding.

Non-Binding Region. The first order condition leads to

$$\frac{z^*}{W} = \frac{(\sigma\sigma^\intercal)^{-1}}{y}(\eta + y\sigma\Sigma). \tag{3}$$

To satisfy the margin constraint, we must have

$$\max_{\lambda\in\Lambda}(\omega^*)^\intercal\lambda \le 1. \tag{4}$$

Given $0 < y < \gamma$, define

$$\lambda^*(y) = \arg\max_{\lambda\in\Lambda}\frac{\lambda^\intercal(\sigma\sigma^\intercal)^{-1}}{y}(\eta + y\sigma\Sigma).$$

Since Λ is a discrete set, $\lambda^*(y)$ exists, is unique and is continuous in y. Clearly, condition (4) is violated for $y > 0$ small enough. Then define

$$y_{N+1,N+1} = \max_{0<y<\gamma}\{(\lambda^*(y))^\intercal\frac{(\sigma\sigma^\intercal)^{-1}}{y}(\eta + y\sigma\Sigma) \ge 1\}.$$

Observe that the map

$$\Upsilon : y \longmapsto (\lambda^*(y))^\intercal\frac{(\sigma\sigma^\intercal)^{-1}}{y}(\eta + y\sigma\Sigma)$$

is a continuous function with $\Upsilon(\gamma) < 1$ (by assumption) and $\lim_{y\to 0}\Upsilon(y) = \infty$. We can conclude that $y_{N+1,N+1}$ exists and is unique, and therefore so is $\lambda^*(y_{N+1,N+1})$. In the sequel, to lighten notation, we shall write λ^* in place of $\lambda^*(y_{N+1,N+1})$, so that

$$y_{N+1,N+1} = \frac{(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\eta}{1-(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\sigma\Sigma},$$

and the margin constraint is not binding for all $y > y_{N+1,N+1}$. Next, recall that we assume that $\gamma^{-1}(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\eta < 1-(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\sigma\Sigma$. If $1-(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\sigma\Sigma < 0$, then we have

$$\frac{(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\eta}{1-(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\sigma\Sigma} > \gamma,$$

i.e. $y_{N+1,N+1} > \gamma$, which is impossible. Hence $0 < \gamma^{-1}(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\eta < 1-(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\sigma\Sigma$. At $y = y_{N+1,N+1}$, the margin constraint starts binding and we can assume that asset allocations given by relationship (3) are all different from zero. Using relationship (3), we obtain the following Hamilton-Jacobi-Bellman equation

$$\left(\theta+(\gamma-1)(m-\gamma\frac{\Sigma^\intercal\Sigma}{2})\right) f(v) = \frac{\gamma}{1-\gamma}\left(f'(v)\right)^{\frac{\gamma-1}{\gamma}} + f'(v) + B^{-1}vf'(v) - \frac{1}{2}\eta^\intercal(\sigma\sigma^\intercal)^{-1}\eta\frac{(f'(v))^2}{f''(v)}. \quad (5)$$

Consider the following change of variables: $x = f'(v), v = -J'(x)$ and $f(v) = J(x) - xJ'(x)$. Using relationship (5), we find that the function J must solve the following linear ODE

$$\begin{aligned}\left(\theta+(\gamma-1)(m-\gamma\frac{\Sigma^\intercal\Sigma}{2})\right) J(x) &= \frac{\gamma}{1-\gamma}x^{\frac{\gamma-1}{\gamma}} + x + (\theta - B^{-1} + (\gamma-1)(m-\gamma\frac{\Sigma^\intercal\Sigma}{2})xJ'(x) \\ &\quad + \frac{1}{2}\eta^\intercal(\sigma\sigma^\intercal)^{-1}\eta x^2 J''(x).\end{aligned}$$

The general solution is

$$J(x) = \frac{\gamma A x^{\frac{\gamma-1}{\gamma}}}{1-\gamma} + Bx + \frac{\gamma K}{\beta-1+\gamma}x^{\frac{\beta-1+\gamma}{\gamma}} + \frac{\gamma L}{\delta-1+\gamma}x^{\frac{\delta-1+\gamma}{\gamma}}, \quad (6)$$

where K and L are constants and β and δ are respectively the positive and negative root of the quadratic[1]

$$\frac{1}{2\gamma^2}\left(\eta^\intercal(\sigma\sigma^\intercal)^{-1}\eta\right)x^2 + \left(A^{-1} - B^{-1} - \frac{1}{2\gamma^2}\eta^\intercal(\sigma\sigma^\intercal)^{-1}\eta\right)x = A^{-1}.$$

Differentiating (6) with respect to x and using the fact that $x = f'(v)$ and $v = -J'(x)$ leads to

$$v + B = Af'(v)^{-\frac{1}{\gamma}} + Kf'(v)^{\frac{\beta-1}{\gamma}} + Lf'(v)^{\frac{\delta-1}{\gamma}}.$$

[1]Note that if x is a root of the quadratic

$$\left(\theta+(\gamma-1)(m-\gamma\frac{\Sigma^\intercal\Sigma}{2})\right) = (\theta - B^{-1} + (\gamma-1)(m-\gamma\frac{\Sigma^\intercal\Sigma}{2})x + \frac{1}{2}\eta^\top(\sigma\sigma^\intercal)^{-1}\eta x^2$$

then $z = \gamma(x-1)+1$ is a root of the quadratic

$$\frac{1}{2}\left(\eta^\top(\sigma\sigma^\intercal)^{-1}\eta\right)x^2 + \left(A^{-1} - B^{-1} - \frac{1}{2}\eta^\top(\sigma\sigma^\intercal)^{-1}\eta\right)x = A^{-1}.$$

Then, when v is large, the margin constraint is irrelevant: asymptotically, the solution $f'(v)$ must be the same as in the unconstrained case, so $f'(v)^{-\frac{1}{\gamma}} \underset{\infty}{\sim} A^{-1}v$. Since $\delta - 1 < 0$, we must have $L = 0$. Finally, K must be positive, otherwise for all v in the non-binding region we have $f'(v) < f_0'(v)$, where f_0 is the unconstrained, reduced, value function. Integrating this relationship from v to $M > v$, we find that

$$f_0(v) < f(v) + f_0(M) - f(M).$$

Since when wealth goes to infinity, constrained and unconstrained value functions coincide, for any given v the previous relationship implies that $f_0(v) < f(v)$, which is impossible.

Binding Region. We now assume that $y \leq y_{N+1,N+1}$. When exactly one constraint among the 2^N linear constraints is binding, the Lagrangian for the maximization problem is

$$L = \omega^\intercal(\eta + y\sigma\Sigma) - \frac{1}{2}y\omega^\intercal\sigma\sigma^\intercal\omega - \psi(\omega^\intercal\lambda^* - 1),$$

where $\psi \geq 0$ is the Lagrange multiplier associated with the constraint. Let ψ_K denote the value of the Lagrange multiplier ψ when exactly K assets are held in the portfolio. The first order condition leads to

$$\frac{z^*}{W} = \frac{(\sigma\sigma^\intercal)^{-1}}{y}(\eta + y\sigma\Sigma - \psi_N\lambda^*). \tag{7}$$

Since the margin constraint is binding, $(\lambda^*)^\intercal\omega^* = 1$, we obtain that

$$\psi_N = \frac{(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\eta - (1 - (\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\sigma\Sigma)y}{(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\lambda^*}. \tag{8}$$

This derivation is valid as long as for all $i = 1, ..., N$, $z_i/\lambda_i \geq 0$. At $y = y_{N+1,N+1}, \psi_N = 0$, the sign of the margin coefficient λ_i must be the same as the sign of $e_i^\intercal(\sigma\sigma^\intercal)^{-1}(\eta + y_{N+1,N+1}\sigma\Sigma)$, for all $i = 1, 2, ..., N$. This provides N conditions to pin down vector λ^*. Exactly one constraint is binding and all asset allocations are different from zero until y becomes small enough. More precisely, from relationships (7) and (8), it is easy to verify that $z_i^* = 0$ exactly when $y = y_{i,N}$ with

$$y_{i,N} = \frac{\left(\lambda^* - \frac{(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\lambda^*}{e_i^\intercal(\sigma\sigma^\intercal)^{-1}\lambda^*}e_i\right)^\intercal(\sigma\sigma^\intercal)^{-1}\eta}{1 - \left(\lambda^* - \frac{(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\lambda^*}{e_i^\intercal(\sigma\sigma^\intercal)^{-1}\lambda^*}e_i\right)^\top(\sigma\sigma^\intercal)^{-1}\sigma\Sigma}.$$

We can assume that $y_{N,N} = \max_{i=1,..,N}\{y_{i,N}\}$ and $y_{N,N} > 0$. When $y = y_{N,N}$, $z_N^* = 0$ and a second linear constraint becomes binding. Hence, we can conclude that for $y_{N,N} < y < y_{N+1,N+1}$, the margin constraint is binding and all assets are optimally held in the portfolio. For all $y < y_{N,N}$, at least two linear constraints are binding and allocation in asset N must be zero. As mentioned earlier, the vectors λ of these two linear constraints have their $N-1$ first components in common and only their last components differ. However, since the allocation in risky asset N is zero and will remain at zero for all $y < y_{N,N}$, these two constraints are actually identical. This implies that for $y < y_{N,N}$, we have to solve the same maximization problem as before when $y_{N,N} < y < y_{N+1,N+1}$, but with $N-1$ risky assets. The maximization problem becomes

$$\max_{\omega}(I_{N-1}\omega)^{\intercal} I_{N-1}(\eta + y\sigma\Sigma) - \frac{y}{2}(I_{N-1}\omega)^{\intercal}(I_{N-1}\sigma\sigma^{\intercal} I_{N-1}^{\top})I_{N-1}\omega$$
$$\text{such that } (I_{N-1}\omega)^{\intercal}(I_{N-1}\lambda^*) \leq 1.$$

Optimal risky allocations are given by

$$I_{N-1}\frac{z^*}{W} = \frac{(I_{N-1}\sigma\sigma^{\intercal} I_{N-1}^{\top})^{-1}}{y} I_{N-1}(\eta + y\sigma\Sigma - \psi_{N-1}\lambda^*), \tag{9}$$

where ψ_{N-1} is the Lagrange multiplier. Using the fact that the constraint is binding; i.e., $(I_{N-1}\omega)^{\intercal}(I_{N-1}\lambda^*) = 1$, we obtain that the value of ψ_{N-1} is given by

$$\psi_{N-1} = \frac{(I_{N-1}\lambda^*)^{\intercal}(I_{N-1}\sigma\sigma^{\intercal} I_{N-1}^{\intercal})^{-1}I_{N-1}\eta - (1 - (I_{N-1}\lambda^*)^{\intercal}(I_{N-1}\sigma\sigma^{\intercal} I_{N-1}^{\intercal})^{-1}I_{N-1}\sigma\Sigma)y}{(I_{N-1}\lambda^*)^{\intercal}(I_{N-1}\sigma\sigma^{\intercal} I_{N-1}^{\intercal})^{-1}I_{N-1}\lambda^*}.$$

Plugging back the value of ψ_{N-1} into relationship (9) leads to

$$\begin{aligned} I_{N-1}\frac{z^*}{W} \quad = \quad & \frac{(I_{N-1}\sigma\sigma^{\intercal} I_{N-1}^{\intercal})^{-1}I_{N-1}(\sigma\sigma^{\intercal} + \sigma\Sigma(\lambda^*)^{\intercal} - \lambda^*(\sigma\Sigma)^{\intercal})I_{N-1}^{\intercal}(I_{N-1}\sigma\sigma^{\intercal} I_{N-1}^{\intercal})^{-1}I_{N-1}\lambda^*}{(I_{N-1}\lambda^*)^{\intercal}(I_{N-1}\sigma\sigma^{\intercal} I_{N-1}^{\intercal})^{-1}I_{N-1}\lambda^*} \\ & + \frac{(I_{N-1}\sigma\sigma^{\intercal} I_{N-1}^{\intercal})^{-1}I_{N-1}\,(\eta(\lambda^*)^{\intercal} - \lambda^*\eta^{\intercal})\,I_{N-1}^{\intercal}(I_{N-1}\sigma\sigma^{\intercal} I_{N-1}^{\intercal})^{-1}I_{N-1}\lambda^*}{(I_{N-1}\lambda^*)^{\intercal}(I_{N-1}\sigma\sigma^{\intercal} I_{N-1}^{\intercal})^{-1}I_{N-1}\lambda^*}\frac{1}{y}. \end{aligned}$$

Without loss of generality, the next threshold value of the lifetime relative risk aversion y at which the next asset is dropped out of the portfolio is $y_{N-1,N-1} = \max_{i=1,..,N-1}\{y_{i,N-1},\, 0 < y_{i,N-1} < y_{N,N}\}$, where

$$y_{i,N-1} = \frac{\left(\lambda^* - \frac{(I_{N-1}\lambda^*)^\intercal (I_{N-1}\sigma\sigma^\intercal I_{N-1}^\intercal)^{-1} I_{N-1}\lambda^*}{e_i^\intercal I_{N-1}^\intercal (I_{N-1}\sigma\sigma^\intercal I_{N-1}^\intercal)^{-1} I_{N-1}\lambda^*} e_i\right)^\intercal I_{N-1}^\intercal (I_{N-1}\sigma\sigma^\intercal I_{N-1}^\intercal)^{-1} I_{N-1}\eta}{1 - \left(\lambda - \frac{(I_{N-1}\lambda^*)^\intercal (I_{N-1}\sigma\sigma^\intercal I_{N-1}^\intercal)^{-1} I_{N-1}\lambda^*}{e_i^\intercal I_{N-1}^\intercal (I_{N-1}\sigma\sigma^\intercal I_{N-1}^\intercal)^{-1} I_{N-1}\lambda^*} e_i\right)^\intercal I_{N-1}^\intercal (I_{N-1}\sigma\sigma^\intercal I_{N-1}^\intercal)^{-1} I_{N-1}\sigma\Sigma},$$

and we assume that $y_{N-1,N-1} > 0$. More generally, for $K < N$ and $i \in \{1,2,...,K\}$, define

$$y_{i,K} = \frac{\left(\lambda^* - \frac{(I_K\lambda^*)^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^*}{e_i^\intercal I_K^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^*} e_i\right)^\intercal I_K^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\eta}{1 - \left(\lambda^* - \frac{(I_K\lambda^*)^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^*}{e_i^\intercal I_K^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^*} e_i\right)^\intercal I_K^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\sigma\Sigma}.$$

Assuming that risky assets can be ordered such that

$$y_{K,K} = \max_{i=1,..,K}\{y_{i,K},\, 0 < y_{i,K} < y_{K+1,K+1}\},$$

with $y_{K,K} > 0$, $K = 2,...N$, we have

$$0 = y_{1,1} < y_{2,2} < y_{3,3} < ... < y_{N,N} < y_{N+1,N+1}.$$

Note that such cutoff values exist since, as we have already shown, for y small enough only one asset is optimally held in the portfolio. Exactly K assets are optimally held in the portfolio, and $N-K+1$ linear constraints among the 2^N possible are binding, when $y_{K,K} < y < y_{K+1,K+1}$. The Lagrange multiplier ψ_K is given by

$$\psi_K = \frac{(I_K\lambda^*)^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\eta - (1 - (I_K\lambda^*)^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\sigma\Sigma)y}{(I_K\lambda^*)^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^*},$$

and risky allocations are given by

$$\begin{aligned} I_K\frac{z^*}{W} \quad = \quad & \frac{(I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K(\sigma\sigma^\intercal + \sigma\Sigma(\lambda^*)^\intercal - \lambda^*(\sigma\Sigma)^\intercal) I_K^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^*}{(I_K\lambda^*)^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^*} \\ & + \frac{(I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\,(\eta(\lambda^*)^\intercal - \lambda^*\eta^\intercal)\, I_K^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^*}{(I_K\lambda^*)^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^*}\frac{1}{y}. \end{aligned}$$

Set

$$L_K = \frac{(I_K\lambda^*)^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\eta}{(I_K\lambda^*)^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^*}$$

$$M_K = \frac{1}{(I_K\lambda^*)^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^*}.$$

Then, we have

$$I_K\frac{z^*}{W} = \frac{(I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\eta}{y} + \left(M_K - \frac{L_K}{y}\right)(I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^*.$$

Using the expressions for z^*/W, we obtain the following, reduced, Hamilton-Jacobi-Bellman equation

$$\begin{aligned}\left(\theta + (\gamma-1)(m - \gamma\frac{\Sigma^\intercal\Sigma}{2})\right) f(v) &= \frac{\gamma}{1-\gamma}\left(f'(v)\right)^{\frac{\gamma-1}{\gamma}} + f'(v) + (B_K^{-1} + \gamma\Sigma^\intercal\Sigma - \gamma\Sigma^\intercal I_K^\intercal I_K\Sigma + L_K)v f'(v) \\ &+\frac{1}{2}\left(\Sigma^\intercal\Sigma + M_K^2(\lambda^*)^\intercal I_K^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^* - 2M_K(\lambda^*)^\intercal I_K^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\sigma\Sigma\right) v^2 f''(v) \\ &-\frac{1}{2}\left(\eta^\intercal I_K^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\eta - L_K^2(\lambda^*)^\intercal I_K^\intercal (I_K\sigma\sigma^\intercal I_K^\intercal)^{-1} I_K\lambda^*\right)\frac{\left(f'(v)\right)^2}{f''(v)}.\end{aligned} \tag{10}$$

Note that the coefficient of the term $\left(f'(v)\right)^2/f''(v)$ is negative if $K > 1$ by the Cauchy-Schwarz inequality, and equal to zero for $K = 1$; the coefficient of the term $v^2 f''(v)$ is equal to $\Sigma^\intercal\Sigma - (I_K\Sigma)^\intercal I_K\Sigma + (M_K(I_K\sigma I_K^\intercal)^{-1} I_K\lambda^* + I_K\sigma\Sigma)^\intercal (M_K(I_K\sigma I_K^\intercal)^{-1} I_K\lambda^* + I_K\sigma\Sigma)$, which is positive.

Deterministic Income and General Preferences.

The Hamilton-Jacobi-Bellman equation for the primal value function F is

$$\theta F = \max_{\frac{z}{W}\in Q} \quad \widetilde{u}(F_1) + (rW+Y)F_1 + mYF_2 + z^\intercal(\mu - r\bar{1})F_1 + \tfrac{1}{2}z^\intercal\sigma\sigma^\intercal z W^2 F_{11},$$

where $\widetilde{u}$ is the convex conjugate of u. This maximization problem is the same as the one solved for the CRRA preferences case so all the results found in the CRRA preference case apply. Furthermore, note that the N conditions that determine the "relevant" vector λ^*, decouple since $\Sigma = 0$ and the coefficients λ_i must have the same sign as $e_i^\intercal(\sigma\sigma^\intercal)^{-1}(\mu - r\bar{1})$.

D. Dual Approach: Fictitious Financial Market

Let a^* and b^* be, respectively, an 1×1 and an $N \times 1$ adapted stochastic process to filtration $\mathbb{F}$ and consider the following fictitious financial market that consists of:

- a riskless bond $\widehat{B}$ with dynamics given by

$$d\widehat{B}_t = (r + a^*)\widehat{B}_t dt,$$

- N risky, non-dividend paying securities whose prices evolve according to:

$$d\widehat{S}_t = I_{\widehat{S}_t}(\mu + b^*)dt + I_{\widehat{S}_t}\sigma dw_t.$$

Dual Formulation

A state price density $\pi^{a,b}$ is an adapted stochastic process to filtration $\mathbb{F}$ defined by $\pi_0^{a,b} = 1$ and

$$d\pi_t^{a,b} = \pi_t^{a,b}\left(-(r + a_t)dt - \left(\sigma^{-1}\left(b_t - a_t\bar{1} + \mu - r\bar{1}\right)\right)^\intercal dw_t\right),$$

where a and b are, respectively, an 1×1 and an $N \times 1$ adapted stochastic process to filtration $\mathbb{F}$.

Effective Domain

For $(a,b) \in \mathbb{R} \times \mathbb{R}^N$, let

$$e(a,b) = \sup_{\frac{z}{z+x} \in Q} - ax - b^\intercal z.$$

The effective domain $\mathcal{N}$ is defined by

$$\mathcal{N} = \left\{(a,b) \in \mathbb{R} \times \mathbb{R}^N, e(a,b) < \infty\right\}.$$

Proposition D.1. *Under the margin constraint, Equation (3) in the paper, the effective domain is given by*

$$\mathcal{N} = \{(a,b) \in \mathbb{R}_+ \times \mathbb{R}_+^N, \kappa^+ a \leq b_i \leq \kappa^- a, i = 1,2,...,N\},$$

and $e(a,b) \equiv 0$, *for all* $(a,b) \in \mathcal{N}$.

Proof. The relationship $e(a,b) \equiv 0$ comes from the fact that Q is a cone. Then, it is easy to see that we must have $a \geq 0$, $b_i \geq 0, i = 1,2,...,N$. If $z_i \geq 0, i = 1,2,...,N$ we have

$$-ax - b^{\mathsf{T}}z = -a\left(x + (1-\lambda^+)\sum_{i=1}^{N} z_i\right) - \sum_{i=1}^{N}(b_i - (1-\lambda^+)a)z_i.$$

Since $z_i \geq 0, i = 1,2,...,N$ we must have $b_i - (1-\lambda^+)a \geq 0, i = 1,2,...,N$. Similarly, when $z_i \leq 0, i = 1,2,...,N$, we have

$$-ax - b^{\mathsf{T}}z = -a\left(x + (1+\lambda^-)\sum_{i=1}^{N} z_i\right) - \sum_{i=1}^{N}(b_i - (1+\lambda^-)a)z_i.$$

Since $z_i \leq 0, i = 1,2,...,N$, we must have $b_i - (1+\lambda^-)a \leq 0, i = 1,2,...,N$. Since $\lambda^+ = \kappa^+ + 1$ and $\lambda^- = \kappa^- - 1$, the desired result follows.

Following the derivation in Cuoco (1997), for *some* suitable price density $\pi^* = \pi^{a^*,b^*}$, the optimization problem, given in Equation (5) in the paper, is equivalent to

$$F(W_0, Y_0) = \max_c E_0\left[\int_0^\infty u(c_s)e^{-\theta s}ds\right] \tag{11}$$

$$\text{such that } E_0\left[\int_0^\infty \pi_s^* c_s ds\right] = W_0 + E_0\left[\int_0^\infty \pi_s^* Y_s ds\right], \tag{12}$$

with $W_0 > 0$ and $Y_0 > 0$ given.

E. Dual Approach

To ensure that the optimization problem, given by Equation (5) in the paper, and (11) are equivalent, it is enough to determine the saddle point $(c^*, \phi^*, (a^*, b^*))$ of the functional

$$\mathcal{L}(c, \psi, (a,b)) = E_0\left[\int_0^\infty u(c_s)e^{-\theta s}ds\right] - \phi\left(E_0\left[\int_0^\infty \pi_s^{a,b}(c_s - Y_s)ds\right] - W_0\right).$$

The maximization over c yields $u'(c_s^*)e^{-\theta s} = \phi\pi_s^{a,b}$ and the Lagrange multiplier ϕ^* is determined by the budget constraint

$$E_0\left[\int_0^\infty \pi_s^{a,b}(I(\phi^*\pi_s^{a,b}e^{\theta s}) - Y_s)ds\right] = W_0,$$

where I is the inverse of the marginal utility function. We define the process $X^{a,b}$:

$$X_t^{a,b} = \phi^*\pi_t^{a,b}e^{\theta t}.$$

The dual value function J is given by

$$J(X_0,Y_0) = \min_{(a,b)\in\mathcal{N}} E_0\left[\int_0^\infty \left(\widetilde{u}(X_s^{a,b}) + X_s^{a,b}Y_s\right)e^{-\theta s}ds\right], \tag{13}$$

where $\widetilde{u}(X) = \max_{c\geq 0} u(c) - Xc$ is the convex conjugate of u. The solution of this minimization problem (a^*,b^*) allows us to recover the state price density $\pi^* = \pi^{a^*,b^*}$. For CRRA preferences, the convex conjugate is given by

$$\widetilde{u}(X) = \begin{cases} \frac{\gamma X^{\frac{\gamma-1}{\gamma}}}{1-\gamma} & , \quad \gamma \neq 1, \\ -\ln X - 1 & , \quad \gamma = 1. \end{cases}$$

Properties of the Dual Value Function

Primal variables (F,W) and dual variables (J,X) are linked by the following Legendre transformation

$$W = -J_1(X,Y) \text{ and } X = F_1(W,Y).$$

As explained in He and Pagès (1993), J is non-increasing and strictly convex in X. It is also easy to check that J is non-decreasing and concave in Y. For the case of a CRRA investor, the dual value function J can be written $J(X,Y) = X^{\frac{\gamma-1}{\gamma}}h(X^{\frac{1}{\gamma}}Y)$, for some smooth function h. Furthermore, J satisfies the following Hamilton-Jacobi-Bellman equation

$$\theta J = \frac{\gamma X^{\frac{\gamma-1}{\gamma}}}{1-\gamma} + XY + (\theta - r)XJ_1 + mYJ_2 + \frac{\Sigma^\intercal\Sigma}{2}Y^2\left(J_{22} + \frac{J_{12}^2}{J_{11}}\right)$$
$$+ \min_{(a,b)\in\mathcal{N}}\left\{-aXJ_1 + \frac{X^2}{2}\left(b+\mu-(r+a)\overline{1} - \frac{\sigma\Sigma YJ_{12}}{XJ_{11}}\right)^\intercal(\sigma\sigma^\intercal)^{-1}\left(b+\mu-(r+a)\overline{1} - \frac{\sigma\Sigma YJ_{12}}{XJ_{11}}\right)J_{11}\right\}.$$

Using the fact that $\gamma X J_{11} = -J_1 + Y J_{12}$ and $-X J_{11}/J_1 = 1/y$, the minimization problem is equivalent to

$$\min_{(a,b)\in\mathcal{N}} a + \frac{1}{2y}\left(\eta + y\sigma\Sigma + b - a\bar{1}\right)^{\intercal}(\sigma\sigma^{\intercal})^{-1}\left(\eta + y\sigma\Sigma + b - a\bar{1}\right). \tag{14}$$

The minimization problem (14) and the maximization problem, given by Equation (7) in the paper, are dual programs of one another: the solution a^* of the dual problem is equal to the Lagrange multiplier ψ of the primal problem. Within the non-binding region, we find that $b_i^* = a^* = 0$. When K assets are optimally held, i.e., $y_{K,K} < y < y_{K+1,K+1}$, the solution of program (14) is

$$\begin{aligned}
a^* &= \psi_K = \frac{(I_K\lambda^*)^{\intercal}(I_K\sigma\sigma^{\intercal}I_K^{\intercal})^{-1}I_K\eta - (1-(I_K\lambda^*)^{\intercal}(I_K\sigma\sigma^{\intercal}I_K^{\intercal})^{-1}I_K\sigma\Sigma)y}{(I_K\lambda^*)^{\intercal}(I_K\sigma\sigma I_K^{\intercal})^{-1}I_K\lambda^*} \\
b_k^* &= (1-\lambda_k^*)a^*,\ k = 1,2,\ldots,K,
\end{aligned}$$

and the fraction of wealth invested in risky assets z^*/W is given by

$$I_K\frac{z^*}{W} = \frac{(I_K\sigma\sigma^{\intercal}I_K^{\intercal})^{-1}}{y}I_K(\eta - \gamma\sigma\Sigma + b^* - a^*\bar{1}).$$

The last $N-K$ constraints of set $\mathcal{N}$ are non binding and the last $N-K$ components of vector b^* are such that $z_k^* = 0$, for $k = K+1, K+2, \ldots, N$.

Remark. If for all $K = 1,\ldots,N$, $1 - \lambda^{\intercal}I_K^{\intercal}(I_K\sigma\sigma^{\intercal}I_K^{\intercal})^{-1}I_K\sigma\Sigma > 0$, which is the case for a low volatility labor income process, it is easy to verify that a^* is a decreasing function of y. This implies that, as the constraint becomes more binding (lower wealth to income ratio), the adjusted risk free rate $r + a^*$ rises, making the bond more attractive to the investor.

Remark. Observe that the right hand side of relationship (11) represents the lifetime resources of the investor. Even though an individual is not allowed to pledge his future labor income in any investment strategy and can only use his financial wealth W_0, his lifetime resources may by far exceed W_0. The margin requirement imposes a limit on the ivnestor's maximum exposure to risky assets. When the margin requirement binds, the investor becomes fairly risk tolerant, which leads him to sacrifice diversification and load up his portfolio with assets that deliver a high expected return.

Remard. For the particular case of deterministic income and independent returns, the investor's choice can be thought of in terms of an adjusted Sharpe ratio for asset k, $\widehat{S}_{P,k}$, defined by

$$\widehat{S}_{P,k} = \frac{\mu_k + b_k^* - (r + a^*)}{\sigma_k}.$$

Inside the non-binding region, for every asset k, the adjusted Sharpe ratio $\widehat{S}_{P,k}$ and the true Sharpe ratio $S_{P,k} = (\mu_k - r)/\sigma_k$ coincide since, when the constraint is not binding, $b_k^* = a^* = 0$. Inside the binding region with N assets, we have $b_k^* = (1 - \lambda_k)a^*$, for $k = 1, 2, \ldots, N$ so indeed

$$\left|\widehat{S}_{P,k}\right| < |S_{P,k}|,$$

since $\mu_k - r$ and λ_k have the same sign. Asset k is dropped out of the portfolio as soon as its adjusted Sharpe ratio $\widehat{S}_{P,k}$ becomes zero. Inside the binding region with only K assets, as the margin constraint becomes more binding, the adjusted Sharpe ratio of the remaining K risky assets shrinks, since a^* rises when y decreases. This result is in line with empirical findings by Ivković, Sialm, and Weisbenner (2008) who report that concentrated portfolios have lower Sharpe ratios.

F. Proof of Proposition I.4

Investment Inside the Non-Binding Region.

We start with some properties of the optimal allocations inside the non-binding region. Consumption, wealth and income are linked by the following relationship $W + BY = Ac + Kc^{1-\beta}Y^{\beta}$ or, equivalently, using reduced variables

$$v + B = A f'(v)^{-\frac{1}{\gamma}} + K f'(v)^{\frac{\beta-1}{\gamma}}. \tag{15}$$

Applying Itô's lemma and identifying the coefficients with the wealth dynamics, the optimal portfolio allocations are given by

$$z^* = z^f - \beta K \frac{(\sigma\sigma^\intercal)^{-1}\eta}{\gamma} f'(v)^{\frac{\beta-1}{\gamma}} Y,$$

where z^f is the unconstrained optimal allocation. When $e_i^\intercal(\sigma\sigma^\top)^{-1}\eta > 0 (< 0)$, the constrained asset allocation z_i^* is lower (higher) than its unconstrained counterpart z_i^f. Next, we show that, inside the non-binding region, income has the same effect on the constrained risky allocations as it has on the unconstrained ones. Differentiating relationship (15) yields

$$\frac{f'(v)}{f''(v)} = -\frac{A}{\gamma} f'(v)^{-\frac{1}{\gamma}} + \frac{\beta - 1}{\gamma} K f'(v)^{\frac{\beta-1}{\gamma}} < 0. \tag{16}$$

From relationships (16) and (15) it is easy to check that the margin requirement is not binding for $f'(v)^{\frac{1}{\gamma}} \leq Z^*$, for some $0 < Z^* < \widehat{Z}$ where $\widehat{Z} = \beta A / ((\beta - 1)B)$. Then, we have

$$\begin{aligned}\frac{\partial z^*}{\partial Y} &= (\sigma\sigma^{\intercal})^{-1}\eta \left(B - \beta K f'(v)^{\frac{\beta-1}{\gamma}} (1 - \frac{\beta-1}{\gamma} \frac{v f''(v)}{f'(v)}) \right) \\ &= \frac{(\sigma\sigma^{\intercal})^{-1}\eta}{A - (\beta-1) K f'(v)^{\frac{\beta}{\gamma}}} \left(AB + (\beta-1)^2 BK f'(v)^{\frac{\beta}{\gamma}} - \beta^2 AK f'(v)^{\frac{\beta-1}{\gamma}} \right).\end{aligned}$$

Set $Z = f'(v)^{\frac{1}{\gamma}}$ and for Z in $[0, Z^*]$, define the auxiliary function h with

$$h(Z) = AB + (\beta-1)^2 BKZ^{\beta} - \beta^2 AKZ^{\beta-1}.$$

h is a smooth function with

$$h'(Z) = \beta(\beta-1)^2 KBZ^{\beta-2}(Z - \widehat{Z}) < 0,$$

so it is decreasing on $[0, Z^*]$, since $Z^* < \widehat{Z}$. We want to show that h is positive on $[0, Z^*]$. First, note that $h(0) = AB > 0$. Then, for $Z = Z^*$, the margin constraint is binding and for $Z \leq Z^*$ we have $(\lambda^*)^{\top} z^* \leq W$ or, equivalently, using the expression of z^*

$$v(1 - \frac{(\lambda^*)^{\intercal}(\sigma\sigma^{\intercal})^{-1}(\mu - r\overline{1})}{\gamma}) \geq \frac{(\lambda^*)^{\intercal}(\sigma\sigma^{\intercal})^{-1}\eta}{\gamma}(B - \beta K f'(v)^{\frac{\beta-1}{\gamma}}).$$

Using relationship (15) we obtain that for all Z in $[0, Z^*]$

$$KZ^{\beta} \geq \varpi(Z - \overline{Z}) \tag{17}$$

$$K(Z^*)^{\beta} = \varpi(Z^* - \overline{Z}), \tag{18}$$

where

$$\overline{Z} = \frac{1 - \frac{(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}(\mu - r\overline{1})}{\gamma}}{1-(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\sigma\Sigma}\frac{A}{B} > 0$$

$$\varpi = \frac{B\left(1-(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\sigma\Sigma\right)}{1-(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\sigma\Sigma + (\beta-1)\frac{(\lambda^*)^\intercal(\sigma\sigma^\intercal)^{-1}\eta}{\gamma}} > 0.$$

Finally, we have

$$h(Z^*) = \frac{B}{Z^*}(\beta\overline{Z} - (\beta-1)Z^*).$$

It remains to show that $Z^* \leq \beta\overline{Z}/(\beta-1)$. Set $x = Z/Z^*$ and $x^* = \overline{Z}/Z^* < 1$, so that for all $0 \leq x \leq 1$, we have $x^\beta \geq (x-x^*)/(1-x^*)$. We want to show that this is the case if and only if $x^* \geq (\beta-1)/\beta$, or, equivalently, $\beta \leq 1/(1-x^*)$. For $x \in [0,1]$, define the auxiliary function f with

$$f(x) = x^\beta - \frac{x-x^*}{1-x^*}.$$

Observe that $f(0) = x^*/(1-x^*) > 0$, $f(1) = 0$ and $f'(x) = \beta x^{\beta-1} - (1-x^*)^{-1}$. If $\beta > 1/(1-x^*)$, then $f'(1) > 0$ and since $f(1) = 0$, it must be the case that $f(1-\varepsilon) < 0$, for some $\varepsilon > 0$ small enough. This leads to a contradiction since by condition (17) f is non-negative on $[0,1]$. Thus, we must have $\beta \leq 1/(1-x^*)$ or, equivalently, $Z^* \leq \beta\overline{Z}/(\beta-1)$. It follows that $h(Z^*) \geq 0$ and for all Z in $[0,Z^*)$, $h(Z) > 0$. We can conclude that z_i^* is increasing (decreasing) with income exactly when $\gamma^{-1}e_i^\top(\sigma\sigma^\top)^{-1}\eta > 0 (< 0)$. Finally, since

$$\frac{z^*}{W} = (\sigma\sigma^\intercal)^{-1}\sigma\Sigma + \frac{(\sigma\sigma^\intercal)^{-1}\eta}{y},$$

we deduce that

$$\frac{\partial}{\partial Y}\left(\frac{1}{y}\right) \geq 0,$$

which implies that

$$\frac{\partial y}{\partial Y} \leq 0.$$

Furthermore, since

$$\frac{\partial y}{\partial Y} = -v\frac{\partial y}{\partial v},$$

we find that

$$\frac{\partial y}{\partial v} \geq 0.$$

At $Y = 0$; i.e., when v is infinite, $y = \gamma$, so we deduce that for all v inside the non-binding region, $y < \gamma$. Finally, note that z^*/W rises as v and W decrease.

Global Properties of the Optimal Consumption c^*.

Recall that $c^* = Y f'(v)^{-\frac{1}{\gamma}}$, so

$$\frac{\partial c^*}{\partial W} = -\frac{f''(v) f'(v)^{-\frac{1}{\gamma}-1}}{\gamma} > 0.$$

Then

$$\frac{\partial c^*}{\partial Y} = \frac{f'(v)^{-\frac{1}{\gamma}}}{\gamma} (\gamma - y),$$

Inside the non-binding region, we have seen that $y < \gamma$, and inside the binding region, we must have $y < y_{N+1,N+1} < \gamma$. Hence, we always have $y < \gamma$ and we conclude that $\partial c^*/\partial Y > 0$.

To prove that the optimal consumption choice for the constrained investor is always lower than the optimal consumption choice for the unconstrained investor, define two new functions

$$A(v) = \frac{(f'(v))^{1-\gamma}}{((1-\gamma) f(v))^{-\frac{1}{\gamma}}} \tag{19}$$

$$B(v) = \frac{(1-\gamma) f(v)}{f'(v)} - v, \tag{20}$$

so that

$$\begin{aligned} f(v) &= \frac{A(v)^{-\gamma}}{1-\gamma} (v + B(v))^{1-\gamma} \\ f'(v) &= A(v)^{-\gamma} (v + B(v))^{-\gamma}, \end{aligned} \tag{21}$$

and $c^* = Y A(v)(v + B(v))$. Differentiating both sides of relationship (21) leads to

$$\frac{f'(v)}{f(v)} = -\gamma \frac{A'(v)}{A(v)} + (1-\gamma) \frac{1 + B'(v)}{v + B(v)},$$

and using relationship (20) yields

$$\gamma\frac{A'(v)}{A(v)} = (1-\gamma)\frac{B'(v)}{v+B(v)}.$$

Next, we show that B is an increasing function of v. Recall that

$$\begin{aligned}
F_{11}(W,Y) =& Y^{-(1+\gamma)} f''(v) \\
F_{12}(W,Y) =& -Y^{-(1+\gamma)}(\gamma f'(v) + v f''(v)) \\
F_{22}(W,Y) =& Y^{-(1+\gamma)}(-\gamma(1-\gamma) f(v) + 2\gamma v f'(v) + v^2 f''(v)).
\end{aligned}$$

On the one hand, by strict concavity of the function F in (W,Y) we find that

$$F_{11}(W,Y)F_{22}(W,Y) - F_{12}^2(W,Y) = -\gamma Y^{-2(1+\gamma)}(\gamma(f'(v))^2 + (1-\gamma) f(v) f''(v)) \geq 0.$$

On the other hand, using Equation (20), we have

$$B'(v) = -\frac{\gamma(f'(v))^2 + (1-\gamma) f(v) f''(v)}{(f'(v))^2}.$$

We conclude that for all $v \geq 0$ $B'(v) \geq 0$. Then, since for large values of v the constrained and the unconstrained problems converge, it follows that $\lim_{v\to\infty} B(v) = B$, and $\lim_{v\to\infty} A(v) = A$), so for all $v \geq 0$, $B(v) \leq B$.

Finally, define the auxiliary function Ψ by

$$\Psi(v) = A(v)\frac{v+B(v)}{v+B} \tag{22}$$

as the ratio of the optimal constrained consumption over the lifetime resources when there are no financial constraints. It follows that

$$\begin{aligned}
\frac{\Psi'(v)}{\Psi(v)} =& \frac{A'(v)}{A(v)} + \frac{1+B'(v)}{v+B(v)} - \frac{1}{v+B} \\
=& \frac{1}{v+B(v)} - \frac{1}{v+B} + \frac{1}{\gamma}\frac{B'(v)}{v+B(v)} \geq 0
\end{aligned}$$

since $B'(v) \geq 0$, and $B(v) \leq B$.

It follows that for all $v \geq 0$, Ψ is a non-decreasing function with $\lim_{v\to\infty} \Psi(v) = A$. Therefore, for all $v \geq 0$, we find that

$$A(v)(v + B(v)) \leq A(v + B),$$

i.e. $c^* \leq c^f$.

G. Proof of Proposition I.5

For $y < y_{N+1,N+1}$, the Hamilton-Jacobi-Bellman equation is such that the coefficient of the term $v^2 f''(v)$ is positive and the coefficient of the term $-(f'(v))^2/f''(v)$ is non-negative. This is exactly the same type of ODE studied by Duffie, Fleming, Soner and Zariphopoulou (1997) for the Merton Problem with unspanned labor income. In Proposition 1 of their paper, these authors establish that $\lim_{v\downarrow 0} f'(v)$ exists, is positive and finite. They also show that $\limsup_{v\downarrow 0} -vf''(v) = 0$. Since $0 < -vf''(v) \leq \sup_{0<x\leq v} -xf''(x)$, it follows that $\lim_{v\downarrow 0} -vf''(v) = 0$. Hence, we have $\lim_{v\downarrow 0} -\frac{vf''(v)}{f'(v)} = 0$. Around $v = 0$, we postulate the following asymptotic expansion

$$f(v) \underset{0}{\sim} d_0 + v - d_1 v^{\frac{3}{2}} + d_2 v^2 + o(v^2),$$

for some constants d_0, $d_1 > 0$ and d_2 to be determined. Our choice for $f'(0) = 1$ is justified because if $f'(0) = 1$, the quantity $\frac{\gamma}{1-\gamma}(f'(v))^{\frac{\gamma-1}{\gamma}} + f'(v)$ achieves its maximum value for $v = 0$. Using the Hamilton-Jacobi-Bellman equation (10) for $K = 1$ and identifying coefficients, we obtain

$$f(0) = d_0 = \frac{1}{(1-\gamma)\left(\theta + (\gamma-1)(m - \gamma\frac{\Sigma^\top\Sigma}{2})\right)} > 0,$$

and

$$\theta + (\gamma-1)(m - \gamma\frac{\Sigma^\intercal\Sigma}{2}) = \frac{9}{8\gamma}d_1^2 + (r - m + \gamma\Sigma^\intercal\Sigma + \frac{\eta_1}{\lambda_1^*}).$$

It follows that

$$d_1 = \frac{2\sqrt{2\gamma(\theta + \gamma(m - (\gamma+1)\frac{\Sigma^\intercal\Sigma}{2})) - (r + \eta_1/\lambda_1^*)}}{3} > 0.$$

This implies that

$$c^* \underset{0}{\sim} Y.$$

Finally notice that

$$y = -\frac{v f''(v)}{f'(v)} \underset{0}{\sim} \frac{3d_1 v^{\frac{1}{2}}}{4}.$$

H. Proof of Proposition I.6

Notice that $u'(c_t^*) = X_t^{a^*,b^*}$ with

$$dX_t^{a^*,b^*} = X_t^{a^*,b^*}(-(r+a_t)dt + (\kappa_t^{a^*,b^*})^\intercal dw_t),$$

where

$$\kappa_t^{a^*,b^*} = -\sigma^{-1}\left(b_t^* - a_t^*\overline{1} + \mu - r\overline{1}\right).$$

Using Itô's lemma, we find that the consumption growth rate is given by

$$\frac{dc_t^*}{c_t^*} = \left(\frac{r+a^*-\theta}{RR(c_t^*)} + \frac{1}{2}\frac{RP(c_t^*)}{(RR(c_t^*))^2}\left\|\kappa_t^{a^*,b^*}\right\|^2\right)dt + \frac{(\kappa_t^{a^*,b^*})^\intercal}{RR(c_t^*)}dw_t,$$

where $RR(c) = -cu''(c)/u'(c)$ is the relative risk aversion ratio and $RP(c) = -cu'''(c)/u''(c)$ is the relative risk prudence ratio. The instantaneous volatility of consumption is given by $\left\|\kappa_t^{a^*,b^*}\right\|^2/(RR(c_t^*))^2$. We now show that for all $t \geq 0$, $\left\|\kappa_t^{a^*,b^*}\right\|^2 \leq \left\|\kappa^{0,0}\right\|^2$. Inside the non-binding region, we have $\kappa_t^{a^*,b^*} = \kappa^{0,0}$. Inside the binding region when K assets are held we have

$$\begin{aligned} b^* &= (\overline{1} - \lambda^*)a^* \\ a^* &= \frac{(I_K\lambda^*)^\intercal(I_K\sigma\sigma^\intercal I_K^\intercal)^{-1}I_K(\mu - r\overline{1}) - y}{(I_K\lambda^*)^\intercal(I_K\sigma\sigma^\intercal I_K^\intercal)^{-1}(I_K\lambda^*)} > 0, \end{aligned}$$

and

$$\left\|\kappa^{a^*,b^*}\right\|^2 = (I_K(\mu - r\overline{1} - \lambda^*a^*))^\intercal(I_K\sigma\sigma^\intercal I_K^\intercal)^{-1}I_K(\mu - r\overline{1} - \lambda^*a^*).$$

Hence

$$\frac{\partial}{\partial y}\left\|\kappa^{a^*,b^*}\right\|^2 = -2\frac{\partial a^*}{\partial y}(I_K\lambda^*)^\intercal(I_K\sigma\sigma^\intercal I_K^\intercal)^{-1}I_K(\mu - r\overline{1} - \lambda^* a^*) = \frac{2}{(I_K\lambda^*)^\intercal(I_K\sigma\sigma^\intercal I_K^\intercal)^{-1}(I_K\lambda^*)} > 0$$

As the margin constraint becomes more binding, y decreases, which reduces the consumption's instantaneous volatility.

I. Numerical Algorithm

A. Model Setup

Market

The continuous-time dynamics of the asset values and income changes are given by Equations (1), (2), and (4) in the paper. We approximate the continuous-time dynamics by a discrete-time Markov chain using the discretization described in He (1990). In this discretization an N dimensional multivariate normal distribution is described by $N+1$ nodes. Discretizing returns in this fashion preserves market completeness in discrete time.

Optimization Problem

We consider the optimization problem described in Equation (5) of Section I in a discrete-time setting, where the investor starts working at time 0 and retires at time T. From the discussion of homogeneity in Section I we can reduce the number of state variables after scaling by income Y_t and obtain the following Bellman equation at $t = 0, \ldots, T-1$:

$$\begin{aligned} f_t(v_t) = \quad & \max_{q_t,\omega_t} u(q_t) + \beta E_t\left[g_t^{1-\gamma} f_{t+1}(v_{t+1})\right] \\ \text{s.t.} \quad & v_{t+1} = g_t^{-1}(v_t + 1 - q_t)\left(\textstyle\sum_{i=1}^N \omega_{i,t} R_{i,t}^e + R^f\right) \\ & \lambda^+ \textstyle\sum_{i=1}^N \omega_{i,t}^+ + \lambda^- \sum_{i=1}^N \omega_{i,t}^- \leq 1 \\ & f_T = \phi_\tau \frac{(v_T+1)^{1-\gamma}}{1-\gamma} \end{aligned} \tag{23}$$

where $v_t = W_t / Y_t$ is the wealth over income ratio; $q_t = c_t / Y_t$ is the consumption over income ratio; $\omega_t = z_t / W_t$ is the portfolio weight; $g_t = Y_{t+1} / Y_t$ is the income growth rate over period t; R^e is the expected one period excess asset return; R^f is the one period return of the money-market account; $f_t(v_t) = Y_t^{-(1-\gamma)} F_t(W_t, Y_t)$ is the reduced value function; and the factor ϕ_τ captures the effect of the investor's remaining lifetime. If the investor's remaining life is τ years, and the opportunity set remains constant, then the factor ϕ_τ is given by

$$\phi_\tau = \left[\frac{1 - (\beta\alpha)^{1/\gamma}}{1 - (\beta\alpha)^{(\tau+1)/\gamma}} \right]^{-\gamma},$$
$$\alpha = E\left[\left(\sum_{i=1}^{N} \omega_i^* R_i^e + R^f \right)^{1-\gamma} \right]$$

where ω^* are the optimal portfolio weights after retirement — see Ingersoll (1987).

B. Solution Methodology

To solve problem (23), we extend the method proposed by Brandt, Goyal, Santa-Clara, and Stroud (2005) to incorporate endogenous state variables and constraints on portfolio weights. We also use an iterative method to find the solution to the Karush-Kuhn-Tucker (KKT) conditions; i.e., the first order conditions with constraints. The idea is to approximate the conditional expectations in the KKT conditions locally within a region that contains the solution to the KKT conditions and iteratively contract the size of the region.

As suggested by Carroll (2006), we separate consumption optimization from portfolio optimization in (23) by defining a new variable, total investment I_t:

$$I_t = v_t - q_t + 1 \tag{24}$$

At the optimal value of consumption, q_t^*, equation (24) defines an one-to-one correspondence between wealth v_t and total investment I_t. Therefore we can specify a particular grid, G, either through wealth, $v_t(G)$, or, equivalently, through investment, $I_t(G)$. Specifying $I_t(G)$ instead of $v_t(G)$ allows splitting problem (23) into two subproblems:

[Portfolio Optimization]

$$
\begin{aligned}
f_t^p(I_t) = \quad & \max_{\omega_t} \beta E_t \left[g_t^{1-\gamma} f_{t+1}(v_{t+1}) \right], \quad t = 0, \ldots, T-1 \\
\text{s.t.} \quad & v_{t+1} = g_t^{-1} I_t \left(\sum_{i=1}^N \omega_{i,t} R_{i,t}^e + R^f \right) \\
& \lambda^+ \sum_{i=1}^N \omega_{i,t}^+ + \lambda^- \sum_{i=1}^N \omega_{i,t}^- \leq 1
\end{aligned} \tag{25}
$$

[Consumption Optimization]

$$
f_t(v_t) = \max_{q_t} u(q_t) + f_t^p(v_t - q_t + 1), t = 0, \ldots, T-1 \tag{26}
$$

where $f^p(\cdot)$ is the value function of the portfolio optimization problem (25). Given the separation of consumption and portfolio optimization, we use the following algorithm to solve problem (23):

Algorithm

Step 1: Set the terminal condition at time T.

Step 2: Find the optimal portfolio and consumption backwards at $t = T-1, T-2, \cdots, 0$:

Step 2.1: Construct a grid for total investment I_t with n_g grid points $\left\{I_t^i\right\}_{i=1}^{n_g}$.

Step 2.2: Find the optimal portfolio and consumption at each grid point $I_t^i, i = 1, \cdots, n_g$:

Step 2.2.1: [Portfolio optimization] given I_t^i, find $\omega_t^*\left(I_t^i\right)$ by solving (25).

Step 2.2.2: [Consumption optimization] given $\left\{I_t^i, \omega_t^*\left(I_t^i\right)\right\}$, find $q_t^*\left(I_t^i\right)$ by solving (26).

Step 2.2.3: Recover state variable v_t at grid point i by $v_t^i = I_t^i + q_t^*\left(I_t^i\right) - 1$.

After specifying the factor ϕ_τ, Step 1 is trivial. Step 2.1 requires constructing a grid in an one-dimensional space. To account for the nonlinearity of the value function at lower wealth levels we place more grid points toward the lower investment values in a double exponential manner as suggested by Carroll (2006).

C. Portfolio Optimization

Given a grid point $I_t^i, i = 1, \cdots, n_g$, we want to optimize over ω_t by solving problem (25). To simplify the problem, and slightly abusing notation, we consider ω_t^+, ω_t^- as choice variables, such that $\omega_t^+ \geq 0, \omega_t^- \geq 0, \omega_t = \omega_t^+ - \omega_t^-$ and solve the following problem:[2]

$$
\begin{array}{ll}
f_t^p(I_t) = & \max\limits_{\omega_t^+, \omega_t^-} \beta E_t \left[g_t^{1-\gamma} f_{t+1}(v_{t+1}) \right] \\
s.t. & v_{t+1} = g_t^{-1} I_t \left[\sum_{i=1}^N \left(\omega_{i,t}^+ - \omega_{i,t^-} \right) R_{i,t}^e + R^f \right] \\
& \lambda^+ \sum_{i=1}^N \omega_{i,t}^+ + \lambda^- \sum_{i=1}^N \omega_{i,t}^- \leq 1 \\
& \omega_{i,t}^+, \omega_{i,t}^- \geq 0, i = 1, \cdots, N
\end{array} \tag{27}
$$

The Lagrangian and KKT conditions of problem (27) are given by:

Lagrangian

$$
\begin{array}{rl}
\mathcal{L}^p \left(\omega_t^+, \omega_t^-, l_t^+, l_t^-, l_t^m \right) = & \beta E_t \left[g_t^{1-\gamma} f_{t+1}(v_{t+1}) \right] + \sum_{i=1}^N l_{i,t}^+ \omega_{i,t}^+ + \sum_{i=1}^N l_{i,t}^- \omega_{i,t}^- \\
& + l_t^m \left(1 - \lambda^+ \sum_{i=1}^N \omega_{i,t}^+ - \lambda^- \sum_{i=1}^N \omega_{i,t}^- \right)
\end{array} \tag{28}
$$

KKT Conditions

$$
\begin{array}{ll}
0 = \beta I_t E_t \left\{ g_t^{-\gamma} \frac{\partial f_{t+1}(v_{t+1})}{\partial v_{t+1}} R_{i,t}^e \right\} + l_{i,t}^+ - l_t^m \lambda^+, i = 1, \ldots, N & \text{FOCs} \\
0 = -\beta I_t E_t \left\{ g_t^{-\gamma} \frac{\partial f_{t+1}(v_{t+1})}{\partial v_{t+1}} R_{i,t}^e \right\} + l_{i,t}^- - l_t^m \lambda^-, i = 1, \ldots, N & \text{FOCs} \\
0 = l_{i,t}^+ \omega_{i,t}^+, i = 1, \cdots, N & \text{Complementarity} \\
0 = l_{i,t}^- \omega_{i,t}^-, i = 1, \cdots, N & \text{Complementarity} \\
0 = l_t^m \left(1 - \lambda^+ \sum_{i=1}^N \omega_{i,t}^+ - \lambda^- \sum_{i=1}^N \omega_{i,t}^- \right) & \text{Complementarity} \\
1 \geq \lambda^+ \sum_{i=1}^N \omega_{i,t}^+ + \lambda^- \sum_{i=1}^N \omega_{i,t}^- & \text{Feasibility} \\
0 \leq \omega_{i,t}^+, \omega_{i,t}^-, l_{i,t}^+, l_{i,t}^-, l_t^m, i = 1, \cdots, N & \text{Feasibility}
\end{array} \tag{29}
$$

where l_t^m is the Lagrange multipliers of the margin constraint; l_t^+ and l_t^- are the Lagrange multipliers of the non-negativity constraints. While in general the KKT conditions are only necessary for optimality,

[2] Notice that to maintain equivalence between (25) and (27) we also need the constraints $\omega_{i,t}^+ \omega_{i,t}^- = 0$ for $i = 1, \cdots, N$, in (27). However, one can show that dropping these constraints will expand the feasible region but will not introduce new optimal solutions which are non-trivially different.

for problem (27) the KKT conditions are both necessary and sufficient since the objective function is concave in $\left(\omega_t^+, \omega_t^-\right)$ and all constraints are linear in $\left(\omega_t^+, \omega_t^-\right)$.

Solving the KKT conditions requires enumeration of all the possibilities for the complementary conditions. In general, the $2N+1$ Lagrange multipliers ($l_t^m, l_{i,t}^+, l_{i,t}^-, i = 1, \cdots, N$) give 2^{2N+1} possible specifications of the complementary conditions. However many of these specifications can be combined or ignored: if the margin constraint is not binding ($l_t^m = 0$) we only need to solve the FOCs without splitting ω_t as $\omega_t^+ - \omega_t^-$; if the margin constraint is binding ($l_t^m > 0$) we can ignore all the specifications with $\omega_{i,t}^+ \omega_t^- > 0, i = 1, \cdots, N$, since these specifications are not optimal. Overall there are $3^N + 1$ specifications that need to be checked. Once a solution to the KKT conditions under any of these specifications is found we can stop since the sufficiency of the KKT conditions guarantees optimality.

Approximation of Conditional Expectations

We use functional approximation to approximate conditional expectations in the KKT conditions as a linear combination of basis functions:

$$E_t \left\{ g_t^{-\gamma} \frac{\partial f_{t+1}(v_{t+1})}{\partial v_{t+1}} R_{i,t}^e \middle| I_t, \omega_t^+, \omega_t^- \right\} \approx \sum_{j=1}^{n_b} \alpha_{ij}(I_t) b_j(\omega_t), i = 1, \cdots, N \tag{30}$$

where n_b is the number of basis functions and $\left\{b_j(\cdot)\right\}_{j=1}^{n_b}$ are the basis functions on portfolio weights $\omega_t = \omega_t^+ - \omega_t^-$. The coefficients $\alpha_{ij}(I_t)$ at each investment grid point $\left\{I_t^i\right\}_{i=1}^{n_g}$ are estimated through cross-test-solution regression in the following way: we randomly generate n_s test solutions $\left\{\omega_t^{(k)}\right\}_{k=1}^{n_s}$ within a set called the test region;[3] for each test solution $\omega_t^{(k)}$ we evaluate the basis functions at the test solution $\left\{b_j(\omega_t^{(k)})\right\}_{j=1}^{n_b}$; given the test solution $\omega_t^{(k)}$ and the investment level I_t, we generate returns for the risky assets following the discretization procedure described in He (1990) and compute the expectation of the left-hand-side of equation (30); the weights $\alpha_{ij}(I_t)$ are estimated by OLS regression across the n_s test solutions.[4]

[3]To guarantee that all the test solutions are feasible we assume that the test region is included in the set of all feasible solutions Q.

[4]The basis functions we use are powers of the choice variables up to third order. We use the multidimensional root-finding solver of the GSL library to solve the KKT conditions. We use 300 grid points and 300 test solutions after checking that the results do not change if 500 grid points and 500 test solutions are used.

Test Region Iterative Contraction (TRIC)

TRIC is a method introduced in Yang (2009) to improve the accuracy of the functional approximation approach for solving the dynamic portfolio choice problem. When we approximate the conditional expectation (30) through cross-test-solution regressions, the quality of the approximation is affected by the number of basis functions n_b, the number of test solutions n_s, and the size of the test region: keeping n_b and n_s constant, the smaller the test region, the more accurate the approximation. This motivates the method of contracting the test region in an iterative manner: at each iteration i, we estimate approximation (30) with test solutions generated within $Q^{(i)}$; using this approximation we solve the KKT conditions to find $\omega^{(i)}$; if $\omega^{(i)} \in Q^{(i)}$ we contract the test region of the next iteration to $Q^{(i+1)} \subset Q^{(i)}$; if the new solution is outside the test region, $\omega^{(i)} \notin Q^{(i)}$, we enlarge the test region of the next iteration to $Q^{(i)} \subset Q^{(i+1)} \subset Q^{(i-1)}$;[5] after each iteration, we check convergence by computing the relative change in portfolio weights $\left\|\omega^{(i)} - \omega^{(i-1)}\right\| / \left\|\omega^{(i-1)}\right\|$, where $\|x\|$ is the the norm of x, defined by $\|x\|^2 = \mathrm{Trace}(x^\intercal x)$, and comparing it with a threshold ε.

To start the procedure we need an initial test region $Q^{(0)}$ that contains the optimal solution. If no further information is available we can set $Q^{(0)} = Q$, the feasible region of problem (27). However, it is possible to obtain a smaller $Q^{(0)}$ if we know the solution for similar parameter values, called a reference solution. We have used our knowledge of the asymptotic behavior of the solutions to construct reference solutions: for each time period we always solve from the grid point with the highest investment level down to the grid point with the lowest investment level; the solution at the higher level grid point serves as the reference solution for the adjacent lower level grid point; when we change between time periods the reference solution at the highest level grid point is set by linearly interpolating the solutions at the next period; at the last time period, $t = T - 1$, the reference solution at the highest level grid point, where the margin constraint is not binding, is set to the analytical solution.

[5] In our numerical tests we contracted the test region by 50%. If the test region did not contain the solution, we expanded the test region by 150%. In the results we report the algorithm converged within two to three iterations for most gridpoints.

D. Consumption Optimization and Value Function Sensitivity

After the optimal portfolio at an investment grid point has been found, we find the optimal level of consumption at that grid point by solving the consumption optimization problem (26). The first order condition leads to

$$q_t^{-\gamma} = \frac{\partial f_t^p(I_t)}{\partial I_t}$$

To evaluate the term $\partial f_t^p(I_t)/\partial I_t$, we apply the envelope theorem to the Lagrangian $\mathcal{L}^p$ in equation (28) and obtain

$$\frac{\partial f_t^p(I_t)}{\partial I_t} = \left.\frac{\partial \mathcal{L}^p}{\partial I_t}\right|_{\omega_t^*(I_t)} = \beta E_t\left[g_t^{-\gamma}\frac{\partial f_{t+1}(v_{t+1})}{\partial v_{t+1}}\left(\sum_{i=1}^{n}\omega_t^*(I_t)R_{i,t}^e + R^f\right)\right]$$

where the conditional expectation is estimated using the discretization scheme for the returns of the risky assets.

In both the portfolio optimization step and the consumption optimization step at time t, we need to evaluate the value function sensitivity $\partial f_{t+1}(v_{t+1})/\partial v_{t+1}$. To evaluate this sensitivity without knowing the functional form of $f_{t+1}(v_{t+1})$, we apply the envelope theorem to the Lagrangian, $\mathcal{L}(q_{t+1}, v_{t+1}) = u(q_{t+1}) + f_{t+1}^p(v_{t+1} - q_{t+1} + 1)$, and get

$$\frac{\partial f_{t+1}(v_{t+1})}{\partial v_{t+1}} = \left.\frac{\partial \mathcal{L}(q_{t+1}, v_{t+1})}{\partial v_{t+1}}\right|_{q_{t+1}^*(v_{t+1})} = \left.\frac{\partial f_{t+1}^p(I_{t+1})}{\partial I_{t+1}}\right|_{q_{t+1}^*(v_{t+1})} = q_{t+1}^{*-\gamma}(v_{t+1})$$

Thus, due to the form of the Lagrangian, the value function sensitivity of problem (23) is completely specified by the optimal consumption as

$$\frac{\partial f_{t+1}(v_{t+1})}{\partial v_{t+1}} = \begin{cases} q_{t+1}^{*-\gamma}(v_{t+1}) & \text{if } t < T-1 \\ \phi_\tau(v_T + 1)^{-\gamma} & \text{if } t = T-1 \end{cases} \tag{31}$$

To evaluate the value function sensitivity at values of v between grid points, we linearly interpolate the optimal consumption results on grid points.

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