Pricing American-Style Options by Monte Carlo Simulation: Alternatives to Ordinary Least Squares

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ABSTRACT

We investigate the performance of the Ordinary Least Squares (OLS) regression method in Monte Carlo simulation algorithms for pricing American options. We compare OLS regression against several alternatives and find that OLS regression underperforms methods that penalize the size of coefficient estimates. The degree of underperformance of OLS regression is greater when the number of simulation paths is small, when the number of functions in the approximation scheme is large, when European option prices are included in the approximation scheme, and when the number of exercise opportunities is large. Based on our findings, instead of using OLS regression we recommend an alternative method based on a modification of Matching Projection Pursuit.

Introduction

Determining the price of American and Bermudan options that depend on a large number of underlying assets is a question of both theoretical and practical interest, and has attracted significant attention in the literature. On the theoretical side, the interest arises because the price of an American option is given by the value function of an optimal stopping problem. Practical interest, on the other hand, is due to the many applications of American options. For example, an application from the area of real options is the estimation of the option value to build a plant and the determination of the optimal time to build. When the option value depends on several underlying stochastic processes, many pricing methodologies become inadequate. Methods that make use of a grid, such as finite difference or finite element methods, or the use of multidimensional lattices, fail due to the curse of dimensionality: computational time increases exponentially with the number of underlying stochastic factors. The only methods whose computational burden remains manageable as the number of underlying stochastic factors increases are based on Monte Carlo simulation.

The apparent difficulty to reconcile Monte Carlo simulation with the dynamic programming framework used to solve optimal stopping problems has been resolved in several ways in the literature. In this paper we concentrate on the function approximation approach introduced by Carriere (1996), where the continuation value function at each possible exercise date is estimated by minimizing squared errors through the use of smooth splines and local regressions. The method was further developed by Tsitsiklis and Van Roy (1999), Tsitsiklis and Van Roy (2001), and Longstaff and Schwartz (2001), who approximate the continuation value function by its projection on the linear span of a set of functions, with the objective of minimizing squared errors.[1]

[1] There are slight differences between the methods: Tsitsiklis and Van Roy (2001) use all the simulated paths to estimate the continuation value, while Longstaff and Schwartz (2001) only use paths that are in-the-money. For further discussion, extensions, as well as links with other methods, see Glasserman (2004).

In this paper we follow the specification presented in Longstaff and Schwartz (2001), and investigate the impact of choosing to minimize squared errors, to the quality of the approximation. While the objective of minimizing squared errors has the advantage that it can be implemented efficiently using Ordinary Least Squares (OLS) regression, it is subject to overfitting and misspecification of the functions used as regressors.[2] As the potential for problems is greater when the number of simulation paths is small, we compare the performance of OLS regression against several alternatives in a set of testcases for a varying number of simulation paths. The alternatives we examine are quantile regression; Tikhonov regularization; Matching Projection Pursuit (MPP); a modified version of Matching Projection Pursuit (MMPP); as well as Classification and Regression Trees (CART), a non-parametric method. All the methods are chosen for their perceived robustness regarding small datasets: quantile regression relies on the estimation of the median rather than the mean, and is less susceptible to a few large fluctuations; the objective of Tikhonov regularization, MPP and MMPP is to minimize squared errors plus a penalty for the size of the coefficients of the linear combination of the functions used to approximate the continuation value function; CART is designed specifically to deal with small datasets.

While OLS regression is the estimation method that produces the estimates with the smallest variance among unbiased estimation methods, our findings demonstrate that in the context of valuing American options, OLS regression is often a poor choice when the number of simulated paths is small. In addition to performing worse than other methods, notably the MMPP method, the performance of OLS regression deteriorates as the number of functions used to approximate the continuation value function increases. Including functions that are likely to be similar to the continuation value function, such as the prices of European options, deteriorates the approximation quality of OLS regression even more. The deterioration in both cases appears to be due to a

[2] Overfitting is a loosely defined term. In the context of our paper, we consider overfitting to be poor out-of-sample performance compared to in-sample performance.

multi-collinearity problem. We find that the quality of the approximation using OLS regression deteriorates as the number of exercise opportunities increases, indicating that bias introduced at an early stage in the backward induction framework propagates and results in an approximate option price far from the true value. A potential explanation to the underperformance of OLS regression is that the estimates computed using Tikhonov regularization, MPP and MMPP have lower variance than the OLS estimates, although they may be biased. The larger variance of OLS regression leads to a rapid deterioration in the OLS estimates, as the results of one estimation are used in a recursive fashion to compute the previous — in terms of exercise dates — estimate.

The rest of this paper is organized as follows: Section I describes the general valuation algorithm and alternative methods to OLS regression. Section II presents the five testcases that we use to evaluate the performance of the different methods. Section III discusses the experimental design of our study and the measures we use to evaluate performance. Section IV presents our computational results. Section V concludes the paper.

I. Valuation Algorithm

A. General Valuation Framework

Pricing American-style options through Monte Carlo simulation involves the general framework of dynamic programming and function approximation. We describe the Least Squares Monte Carlo (LSM) algorithm below and in Figure 1. Further details can be found in Longstaff and Schwartz (2001).

The algorithm involves the generation of paths for the values of the state variables by Monte Carlo simulation.[3] We use the following notation: $S_t^{(i)}$ is the value of the state variables at time t along path i; h is the option payoff; V is the option value; and $\{t_i\}_{i=0}^N$ are the possible exercise times. The algorithm approximates the continuation value of the option at each possible exercise time by a linear combination of a set of basis functions $\{\phi_j\}_{j=1}^{N_b}$.[4]

LSM Algorithm

Step 1: generate M paths for the values of the state variables at all possible exercise times

Step 2: at the terminal time t_N, set the option value V equal to the payoff

$$V\left(S_{t_N}^{(m)}, t_N\right) = h\left(S_{t_N}^{(m)}, t_N\right), m = 1, ..., M$$

Step 3: for the set of paths $\{i_l\}_{l=1}^L$, for which the option is in-the-money; i.e., $h(S_{t_{N-1}}^{i_l}, t_{N-1}) > 0$, find coefficients $a_j^*(t_{N-1})$ to minimize the norm

$$\left\| \sum_{j=1}^{N_b} a_j\left(t_{N-1}\right) \begin{pmatrix} \phi_j(S_{t_{N-1}}^{(i_1)}) \\ \phi_j(S_{t_{N-1}}^{(i_2)}) \\ \vdots \\ \phi_j(S_{t_{N-1}}^{(i_L)}) \end{pmatrix} - e^{-r(t_N - t_{N-1})} \begin{pmatrix} V(S_{t_N}^{(i_1)}, t_N) \\ V(S_{t_N}^{(i_2)}, t_N) \\ \vdots \\ V(S_{t_N}^{(i_L)}, t_N) \end{pmatrix} \right\|$$

[3] The state variables include the prices of the assets, but may also include the values of other stochastic processes, such as the interest rate, the volatility and the running maximum or minimum of the price of an asset.

[4] We are abusing terminology by calling the set of functions $\{\phi_j\}_{j=1}^{N_b}$ a basis. Ideally the linear span of the functions $\{\phi_j\}_{j=1}^{N_b}$ would include the continuation value. Unfortunately this can not be guaranteed in the general case.

Step 4: for each path update the value function at time t_{N-1}

$$V\left(S_{t_{N-1}}^{(m)}, t_{N-1}\right) = \begin{cases} h\left(S_{t_{N-1}}^{(m)}, t_{N-1}\right), & \text{if } h\left(S_{t_{N-1}}^{(m)}, t_{N-1}\right) \geq \sum_{j=1}^{N_b} a_j^*\left(t_{N-1}\right)\phi_j\left(S_{t_{N-1}}^{(m)}\right) \\ e^{-r(t_N - t_{N-1})}V\left(S_{t_N}^{(m)}, t_N\right), & \text{otherwise} \end{cases}$$

Step 5: repeat Steps 3 and 4 for possible exercise times $t_{N-2}, t_{N-3}, ...,$ until time t_0.

The inputs of the LSM algorithms are the number of Monte Carlo paths M, the basis functions $\{\phi_j\}_{j=1}^{N_b}$, and the vector norm $\|\cdot\|$. Longstaff and Schwartz (2001) use polynomials for the basis functions and the L_2 vector norm, which leads to OLS regression.

B. Convergence

The convergence properties of the LSM algorithm have been studied by Clement, Lamberton, and Protter (2002). There are two types of approximations in the algorithm.

- Type I: approximate the continuation value by its projection on the linear space of a finite set of basis functions.
- Type II: use Monte Carlo simulations and OLS regression to estimate the coefficients of the basis functions.

Clement, Lamberton, and Protter (2002) showed that under certain regularity conditions on the continuation value, Type II approximation error decreases to zero as the number of simulation paths goes to infinity, holding the number of basis functions fixed (fixed Type I approximation error). In addition, if the Type II approximation is exact; i.e., the coefficients of the projection can be calculated without random errors, then the Type I approximation error diminishes as polynomials of the state variables of increasing degree are added to the basis functions. As the degree of the polynomials tends to infinity, Type I error tends to zero; i.e., the algorithm converges to the true price as the number of simulation paths and basis functions tends to infinity. If the linear span of the basis functions does not include the continuation value function; i.e., the Type I approximation is not exact, then there will always be an error in the overall approximation, irrespective of the number of simulation paths.

Glasserman and Yu (2004) study the convergence rate of the algorithm when the number of basis functions and the number of paths increase simultaneously. They demonstrate that in certain cases, in order to guarantee convergence, the number of paths must grow exponentially with the number of polynomial basis functions when the underlying state variable follows Brownian motion, or faster than exponential when the underlying state variable follows geometric Brownian motion.

C. Ordinary Least Squares and Alternatives

Among possible choices for the norm used in the minimization in Step 3 of the LSM algorithm, the L_2 norm has the advantage of being easy to implement, as it corresponds to OLS regression. In this section, we discuss the benefits and drawbacks of OLS regression and describe alternative norms and algorithms.

C.1. Ordinary Least Squares Regression

Given observations $\{y_i\}_{i=1}^{L}$ and a set of regressors $\{x_i\}_{i=1}^{N_b}$, OLS regression finds the coefficients $\{a_i\}_{i=1}^{N_b}$ that minimize the sum of squared errors:

$$\min_a \left[\sum_{i=1}^{L} (y_i - \hat{y}_i)^2 \right] = \min_a \left[\sum_{i=1}^{L} \left(y_i - \left(\sum_{j=1}^{N_b} a_j x_j \right)_i \right)^2 \right]$$

where $(\ )_i$ corresponds to the i^{th} component of a vector. In the context of this paper, the observed values correspond to discounted option values from the next possible exercise time; i.e., $y_i = e^{-r(t_{j+1}-t_j)}V(S^{(i)}_{t_{j+1}}, t_j)$. The regressors correspond to the basis functions $\{\phi_i\}_{i=1}^{N_b}$.

OLS regression has important optimality properties. For example, among all linear, unbiased estimators, OLS regression is guaranteed to produce estimates of the coefficients $\{a_i\}_{i=1}^{N_b}$ with the smallest variance (Gauss-Markov theorem, see Greene (2000)). For the purpose of pricing American-style options, using OLS regression can guarantee the convergence of the computed price to the true option price as the number of simulation paths and the number of basis functions go to infinity. However, it is possible that biased estimators may produce estimates of the coefficients with smaller variance. The problem of large variance in the estimation of the coefficients is particularly severe when the number of regressors is large compared to the number of observations, and when the regressors are highly correlated. Large variance in the estimates manifests itself as poor out-of-sample performance, and is often loosely described as overfitting. In our context, the problem is potentially more severe, since OLS estimates for one exercise date influence the option values for prior exercise dates in a recursive fashion.

C.2. Quantile Regression – Least Absolute Error Regression

Quantile regression is a statistical method that estimates conditional quantile functions. Just as OLS regression estimates the conditional mean, quantile regression estimates the conditional median, as well as the full range of other conditional quantile functions. In this paper, we use the median (50% quantile) regression to approximate the continuation value. This is equivalent to the Least Absolute Error regression where one minimizes the sum of absolute errors; i.e.,

$$\min_a \left[\sum_{i=1}^{L} |y_i - \hat{y}_i|\right] = \min_a \left[\sum_{i=1}^{L} \left| y_i - \left(\sum_{j=1}^{N_b} a_j x_j\right)_i \right|\right]$$

The advantage of quantile regression is that the estimation is less likely to be influenced by outliers, producing smaller variation in the coefficient estimates compared to OLS regression.

Quantile regression software is available in most modern statistical software packages. In this paper we use the "`quantreg`" package in R. More details about quantile regression and the "`quantreg`" package can be found in Koenker and Hallock (2001).

C.3. Tikhonov Regularization

Tikhonov regularization is a regularization method developed by Phillips (1962) and Tikhonov (1963) to treat ill-posed problems with nearly linearly dependent predictors. The method involves a trade-off between the "size" of the regularized solution and the "quality" of the solution in terms of fitting the given data. Tikhonov regularization can be formulated as:

$$\min_a \left[\sum_{i=1}^{L} \left(y_i - \left(\sum_{j=1}^{N_b} a_j x_j \right)_i \right)^2 + \lambda^2 \sum_{i=1}^{L} \left(\sum_{j=1}^{N_b} L_{ij} \left(a_j - \bar{a}_j \right) \right)^2 \right]$$

where y is the vector of observed values, x is a matrix whose columns correspond to the predictors, a is the vector of the coefficients of the predictors, λ is a regularization parameter that determines the trade-off between the size of the solution measured by $\sum_{i=1}^{L} \left(\sum_{j=1}^{N_b} L_{ij} \left(a_j - \bar{a}_j \right) \right)^2$, and the quality of the solution measured by $\sum_{i=1}^{L} \left(y_i - \left(\sum_{j=1}^{N_b} a_j x_j \right)_i \right)^2$. The vector $\bar{a}$ is a prior estimate of a.[5] L is a weight matrix. In this paper, we set $\bar{a} = 0$ and $L = I$, the identity matrix.

The choice of the regularization parameter λ is crucial and also case dependent. To determine the "best" λ, a graphical tool called L-curve is often used to quantify the

[5] Smart choices of $\bar{a}$ can be helpful in improving the results, but hard to find with low computational cost. Notice that when $\lambda = 0$, or when $\bar{a}$ is equal to the OLS regression estimate, Tikhonov regularization is equivalent to OLS regression. We have used Matching Projection Pursuit (described in the next section) to generate $\bar{a}$, but did not see any improvement in the results.

trade-off between size and quality. A typical L-curve is shown in Figure 2. The "optimal" regularization parameter λ^* corresponds to the corner point in the figure. Choosing $\lambda > \lambda^*$ tends to over-control the size and leads to poor quality of data fitting. On the other hand, choosing $\lambda < \lambda^*$ leads to better fitting of the data, but with rapidly increasing solution size, which is prone to overfitting. More information about the L-curve and its applications can be found in Hansen (2001). We use the Regularization Tools package in Matlab, developed by Hansen (1994), to perform Tikhonov regularization.

C.4. Matching Projection Pursuit

The projection pursuit method was developed by Kruskal (1969) to interpret high-dimensional data with well-chosen lower-dimensional projections. The idea is to recursively identify optimal projection directions with respect to minimizing residuals in a hierarchical fashion, one direction at a time. A particular version of projection pursuit, Matching Projection Pursuit (MPP), introduced by Mallat and Zhang (1993), only searches within a given set of directions, called a dictionary. In our context, the dictionary is the set of basis functions. MPP can be considered as a greedy algorithm of finding the optimal projection of a vector onto the linear span of a set of, possibly, non-orthogonal vectors. Each iteration chooses among the set of available projection directions the direction that minimizes the L_2 norm of the residual vector. The chosen direction is the one that has the largest inner product, among all the possible directions, with the vector being projected. Given the inner product interpretation, it is possible to implement MPP efficiently. The process can be written as an algorithm:

MPP Algorithm

Step 1: Set up a dictionary with basis functions: $\{x_j\}_{j=1}^{N_b}$

Step 2: Determine the cutoff c

Step 3: Among the dictionary directions, find the direction $\gamma^{(1)}$ that best describes the data $y^{(0)}$.

$$\gamma^{(1)} = \left\{ h : \frac{\left|\left\langle y^{(0)}, x_h \right\rangle\right|}{\|x_h\|_2} \geq \frac{\left|\left\langle y^{(0)}, x_j \right\rangle\right|}{\|x_j\|_2}, \forall j \right\}$$
$$a_1 = \frac{\left\langle y^{(0)}, x_{\gamma^{(1)}} \right\rangle}{\left\|x_{\gamma^{(1)}}\right\|_2^2}$$

where $\left\langle y^{(0)}, x_h \right\rangle = \sum_{i=1}^{L} y_i^{(0)} (x_h)_i$ and $\|x_h\|_2 = \sqrt{\sum_{i=1}^{L} (x_h)_i^2}$.

Step 4: Subtract all the information along the best direction from the data and compute the residual vector $y^{(1)}$.

$$y^{(1)} = y^{(0)} - a_1 x_{\gamma^{(1)}}$$

Step 5: Check the percentage reduction of the norm of the residual vector. If $\frac{\|y^{(0)}\|_2 - \|y^{(1)}\|_2}{\|y^{(0)}\|_2} < c$, stop; otherwise go to Step 3 and repeat using the residual vector $y^{(k)}$.

$$\gamma^{(k+1)} = \{h : \frac{\left|\left\langle y^{(k)}, x_h \right\rangle\right|}{\|x_h\|_2} \geq \frac{\left|\left\langle y^{(k)}, x_j \right\rangle\right|}{\|x_j\|_2}, \forall j\}$$
$$a_{k+1} = \frac{\left\langle y^{(k)}, x_{\gamma^{(k+1)}} \right\rangle}{\left\|x_{\gamma^{(k+1)}}\right\|_2^2}$$
$$y^{(k+1)} = y^{(k)} - a_{k+1} x_{\gamma^{(k+1)}}$$

One of the inputs in the MPP algorithm is the choice of the cutoff in Step 2. We choose the cutoff corresponding to the explanatory power of the dictionary with respect to a random vector: we randomly generate 300 vectors with the same length as the data; for each random vector we estimate the percentage reduction in size by finding the best projection with respect to the dictionary. The cutoff value is set to the average percentage reduction in the size of the random vector over the 300 random vectors. We chose 300 as the sample size of the random vectors so that the determination of the cutoff is stable.

C.5. Modified Matching Projection Pursuit

The Modified Matching Projection Pursuit (MMPP) method constrains the magnitude of the projection in each iteration to a small quantity rather than the magnitude that would minimize the L_2 norm of the residual vector.

MMPP Algorithm

Step 1: Set up a dictionary with basis functions: $\{x_j\}_{j=1}^{N_b}$

Step 2: Determine the cutoff c.

Step 3: Determine the step size ϵ.

Step 4: Among the dictionary directions, find the direction $\gamma^{(1)}$ that best describes the data $y^{(0)}$.

$$\begin{aligned}\gamma^{(1)} =& \left\{h : \frac{\left|\left\langle y^{(0)}, x_h\right\rangle\right|}{\|x_h\|_2} \geq \frac{\left|\left\langle y^{(0)}, x_j\right\rangle\right|}{\|x_j\|_2}, \forall j\right\} \\ a_1 =& \frac{\left\langle y^{(0)}, x_{\gamma^{(1)}}\right\rangle}{\left\|x_{\gamma^{(1)}}\right\|_2^2}\end{aligned}$$

where $\left\langle y^{(0)}, x_h\right\rangle = \sum_{i=1}^{L} y_i^{(0)} (x_h)_i$ and $\|x_h\|_2 = \sqrt{\sum_{i=1}^{L} (x_h)_i^2}$.

Step 5: Project along the best direction under the step size constraint and compute the residual vector $y^{(1)}$.

$$\begin{aligned}a_1' =& \text{sign}(a_1) \times \min\{|a_1|, \epsilon\} \\ y^{(1)} =& y^{(0)} - a_1' x_{\gamma^{(1)}}\end{aligned}$$

Step 6: Check the percentage reduction of the norm of the residual vector. If $\frac{\|y^{(0)}\|_2 - \|y^{(1)}\|_2}{\|y^{(0)}\|_2} < c$, stop; otherwise go to Step 3 and repeat using the residual vector $y^{(k)}$.

$$
\begin{aligned}
\gamma^{(k+1)} =& \{h : \frac{\left|\left\langle y^{(k)}, x_h \right\rangle\right|}{\|x_h\|_2} \geq \frac{\left|\left\langle y^{(k)}, x_j \right\rangle\right|}{\|x_j\|_2}, \forall j\} \\
a_{k+1} =& \frac{\left\langle y^{(k)}, x_{\gamma^{(k+1)}} \right\rangle}{\left\|x_{\gamma^{(k+1)}}\right\|_2^2} \\
a'_{k+1} =& \text{sign}\left(a_{k+1}\right) \times \min\left\{\left|a_{k+1}\right|, \epsilon\right\} \\
y^{(k+1)} =& y^{(k)} - a'_{k+1} x_{\gamma^{(k+1)}}
\end{aligned}
$$

The step size, ϵ, controls how aggressive the algorithm is. For a very large step size, the MMPP method is equivalent to the MPP method. We set the step size to 1% of the absolute value of the coefficient along the best direction found in the first round of searching; i.e., $\epsilon = 1\% \times |a_1|$. The cutoff value is computed similarly to the case of the MPP method.

The link between shrinkage estimators and the Tikhonov and MMPP methods

An alternative to OLS regression that we are not directly investigating involves the use of shrinkage estimators. The best known ones are ridge regression and Least Absolute Shrinkage and Selection Operator (LASSO). In both cases the objective is to minimize the sum of squared errors subject to a constraint on the size of estimated coefficients. In the case of ridge regression the constraint involves the sum of the squares of the estimated coefficients, while in the case of LASSO it involves the sum of the absolute values of the estimated coefficients. There are free parameters in each method that correspond to the magnitude of the constrained sums. There is a direct link between ridge regression and Tikhonov regularization, and between LASSO and MMPP. The choice of the magnitude of the constrained sums is similar to the choice of the parameter λ in Tikhonov regularization and to the choice of the cutoff in MMPP. Hastie, Tibshirani, and Friedman (2001) describe the ridge regression and LASSO methods in detail, and provide a numerical algorithm for estimating the regression coefficients of LASSO that is equivalent to the MMPP method.

C.6. Classification and Regression Tree Method – CART

The Classification and Regression Tree (CART) method was developed by Breiman, Friedman, Olshen, and Stone (1984). It is a nonparametric method that can be used to predict values of continuous dependent variables (regression problem) or classify categorical variables (classification problem). The method makes no assumptions on the relationships between the predictors and the dependent variables. The main idea of CART is to partition the space spanned by the predictors into a set of rectangles, and then fit a simple model in each rectangle. The process of partitioning is performed recursively to balance between the size of the tree and the quality of data fitting. The intuition is that a very large tree might overfit the data, while a small tree might overlook important details in the data. Because of its nonparametric nature, CART is well suited for problems where there is little a priori knowledge regarding the relationship between predictors and dependent variables. This is the situation we face when pricing American-style options by Monte Carlo Simulation. The continuation value is unknown, problem dependent, nonparametric, and nonlinear. We implement the regression tree method using the "`rpart`" package in R.[6]

II. Testcases

To test the different approximation methods, we have employed a set of five testcases that were introduced by Fu, Laprise, Madan, Su, and Wu (2001) as a benchmark for

[6] In addition, we have examined Bayesian Additive Regression Trees (BART) on several testcases. We found no improvement in the results and the computational time is much longer than CART. We do not report the results based on BART in this paper.

evaluating the performance of Monte Carlo based numerical methods. Before describing the testcases, we introduce some notation.

T :	expiration date
t_i :	possible exercise time, $i = 0, ...N$
K :	strike price
r :	interest rate
σ :	volatility
S_t :	stock price at time t
S_t^j :	stock price at time t of stock j, $j = 1, ..., n$
h_t :	payoff if the option is exercised at time $t \in \{t_i\}_{i=0}^N$

Testcase 1: Call Option with Discrete Dividends

Testcase 1 is an American call option on a single stock that at times t_i, $i = 1, 2, ...N$, distributes discrete dividends D_i. The payoff function is given by:

$$h_t = (S_t - K)^+$$

The dividend D_i can be deterministic or stochastic. The ex-dividend stock price drops by the amount of the dividend; i.e.,

$$S_{t_i^+} = S_{t_i} - D_i, i = 1, ..., N$$

The stock price process net of the present value of dividends, $\tilde{S}_t$, follows geometric Brownian motion under the risk-neutral measure

$$d\tilde{S}_t = \tilde{S}_t \left[rdt + \sigma dZ_t \right]$$

The actual stock price process is given by

$$S_t = \tilde{S}_t + \sum_{j=i+1}^{N} D_j e^{-r(t_j - t)},\ t \in [t_i, t_{i+1}), i = 0, 1, ...N$$

Under the assumption of a frictionless market the optimal exercise policy of a call option is to always exercise right before an ex-dividend date or at the expiration date. Thus, the potential early exercise dates are the ex-dividend dates $\{t_i\}_{i=1}^N$.

Testcase 2: Call Option with Continuous Dividends

Testcase 2 is an American call option on a single stock with continuous dividends paid at a rate δ and discrete possible exercise times $\{t_i\}_{i=0}^N$. In this case the dividends are directly embedded into the stock price dynamics. We assume that the stock price S_t follows geometric Brownian motion under the risk-neutral measure

$$dS_t = S_t \left[(r - \delta)\, dt + \sigma dZ_t \right]$$

The payoff function is the same as Testcase 1.

Testcase 3: American-Asian Call Option

Testcase 3 is an American-Asian call option with payoff defined by

$$h_t = \left(\bar{S}_t - K \right)^+$$

where

$$\bar{S}_t = \frac{1}{n_t + 1} \sum_{j=0}^{n_t} S_{t_j}$$

and

$$t_j = t' + (t - t') \frac{j}{n_t}$$

$\bar{S}_t$ is the discrete average of the stock prices where averaging starts at a pre-specified date t' up to the exercise time t. Similar to Fu, Laprise, Madan, Su, and Wu (2001), we used daily averaging starting on day t'. $n_t + 1$ prices are averaged in calculating $\bar{S}_t$. We allow early exercise at times $\{t_i\}_{i=0}^N$. The underlying stock price process follows geometric Brownian motion with continuous dividends, just as Testcase 2.

Testcase 4: Put Option on a Jump-Diffusion Asset

Testcase 4 is a put option defined on a single underlying stock with discrete exercise times. The payoff function is given by

$$h_t = (K - S_t)^+, t \in \{t_i\}_{i=0}^N$$

The stock price process follows jump-diffusion:

$$S_t = S_0 \exp\left[\left(r - \sigma^2/2\right) t + \sigma\sqrt{t} Z_0 + \sum_{j=1}^{N(t)} \left(\delta Z_j - \delta^2/2\right)\right]$$

where $Z_j \sim N(0,1), j = 0, ..., N(t)$, are independent, identically distributed (i.i.d.) and $N(t) \sim \text{Poisson}(\lambda t)$. Thus, the jump sizes are i.i.d. random variables that are lognormally distributed with mean $\mu_j = -\delta^2/2$ and standard deviation $\sigma_j = \delta$. With this choice, the expected jump sizes are equal to one.

Testcase 5: Max-Call Option on Multiple Underlying Assets

Testcase 5 is a call option on the maximum of n stocks. The payoff function is given by:

$$h_t = \left[\left(\max_{j=1,\ldots,n} S_t^j\right) - K\right]^+$$

Early exercise is possible at discrete times $\{t_i\}_{i=0}^N$. The underlying assets are assumed to follow correlated geometric Brownian motions with continuous dividends

$$dS_t^j = S_t^j\left[(r-\delta_j)\,dt + \sigma_j dZ_t^j\right], j = 1,...n$$

where Z_t^j is a standard Brownian motion and the instantaneous correlation between Z^j and Z^k is ρ_{jk}. Similar to Fu, Laprise, Madan, Su, and Wu (2001), we assume $\rho_{jk} = \rho$ for all $j, k = 1, ..., n$ and $j \neq k$.

III. Experimental Design

As we mentioned in Section I.B, Clement, Lamberton, and Protter (2002) have shown that the LSM algorithm using OLS regression converges to the true option price as the number of simulation paths and the number of basis functions go to infinity. In practice, both the number of basis functions and the number of simulation paths are finite, and potentially small. In this paper, we focus on two questions:

1) What is the behavior of the different estimation methods, for a given set of basis functions and a small number of simulation paths?
2) Given a small number of simulation paths, how does the behavior of the different estimation methods depend on the choice of basis functions?

To address these questions, we have designed a numerical experiment. We study the quality of the price calculated using the LSM algorithm with different numbers of simulation paths and different sets of basis functions for the testcases described in Section II using different estimation methods. The experimental design was chosen to address several challenges, outlined below.

- The true price is unknown: The first problem in quantifying the quality of an estimation method is that the prices of the options in the testcases we study are unknown. However, once a set of basis functions is chosen, the LSM algorithm cannot do better than approach the option value computed in the limit of an infinite number of simulation paths. To reflect this limitation, we compare the approximate option prices obtained by the different estimation methods, to the option price obtained using OLS regression with a very large number of paths. In practice, we determine the limiting price by ensuring that doubling the number of paths no longer influences the option price.

- The computed price for each estimation method is a random variable: Since simulation is the basis of the LSM algorithm, the resulting approximate option prices are random variables. In order to compare between different estimation methods, we use multiple runs to obtain a distribution of the estimated option price. From the distribution of the estimated option price, we calculate the mean and the standard error of the mean. To avoid any potential bias, we use independent sets of paths for each estimation method.

- Choice of basis functions: An important part of the algorithm is the choice of basis functions. So far in the literature the choice has been to use polynomials, see for example Tsitsiklis and Van Roy (2001), and Longstaff and Schwartz (2001). Moreno and Navas (2003) found that different types of polynomials; e.g., Chebyshev, Hermite, Laguerre, and Legendre, lead to very small variations in the option value when the largest degree of the different types of polynomials is the same and 100,000 paths are used. Given our focus on evaluating the performance of different estimation methods, we investigate bases that consist of polynomials of different degrees, as well as bases that include functions corresponding to European option prices with different strikes and maturities. Our motivation for including such functions is that they are likely to be more similar to the continuation value than polynomials. While one would expect that including basis functions that are more similar to the function being estimated would improve the approximation, it is possible that the opposite is true, especially since European option prices with different strikes and maturities are highly correlated.

- Computational efficiency: A concern when using alternatives to OLS regression is the potential additional computational cost. The methods we have chosen to study, while less familiar than OLS regression, are nonetheless similar in terms of computational efficiency: quantile regression reduces to a linear optimization problem; similarly to OLS the computational time of Tikhonov regularization increases linearly with the number of observations; MPP and MMPP also scale linearly with the number of observations as they reduce to matrix-vector multiplication, where the matrix has a number of rows equal to the number of basis functions and a number of columns equal to the number of simulation paths. In numerical experiments we have confirmed that CART also scales linearly with the number of simulation paths. In our computations there were very small differences in computational time between OLS regression and the alternative estimation methods.

IV. Computational Results

A. Setup

Following the design outlined in Section III, we estimate the asymptotic approximation to the option price for each of the testcases using 100,000 simulation paths and OLS regression. To evaluate the estimation methods, we compare the values obtained for different sets of basis functions and different numbers of simulation paths. We use samples of 100, 1,000, 5,000, and 10,000 simulation paths and basis functions that include polynomials of different degrees, as well as a mix of polynomials and European option prices in cases where a closed form solution for European option prices exists. Our code is implemented in Matlab, including path simulation, general valuation algorithm, input-output, and several projection methods (OLS regression, MPP, and MMPP method). The other projection methods are implemented through existing software packages: Tikhonov regularization by the Regularization Tools package of Matlab, quantile regression by the "`quantreg`" package of R, and regression tree method by the "`rpart`" package of R. We use antithetic paths to reduce the variance of the estimates.

In Testcases 1, 2 and 4 the basis functions include polynomials of the underlying stock price, while in the case of the average option (Testcase 3) they include polynomials of the running average of the stock price. The column "N_b" in the tables corresponds to the highest degree of polynomial used in the basis. In the case of the max-call on several assets (Testcase 5), in addition to polynomials of the underlying stock price we also used the largest and the second largest stock price as predictors. In this case, $N_b = 0$ corresponds to a basis with the constant function, the largest stock price $S_{(n)}$, and the second largest stock price $S_{(n-1)}$; $N_b = 1$ adds the linear functions $S_1,\ S_2,\ \cdots,\ S_n$; $N_b = 2$ adds the diagonal quadratic terms $S_1^2,\ S_2^2,\ \cdots,\ S_n^2$; $N_b = 3$ adds the off-diagonal quadratic terms; $N_b = 4$ adds the diagonal cubic terms $S_1^3,\ S_2^3,\ \cdots,\ S_n^3$; $N_b = 5$ adds the off-diagonal cubic terms. For example, in the case of the max-call option on three assets, the basis functions used for each value of "N_b" are listed in Table I.

When European options are included in the basis (Testcases 1, 2, 3, 4 and Testcase 5 with two assets), there are nine functions added to the set of basis functions: the price of the European options with strikes at-the-money, 10% in-the-money, and 10% out-of-the-money, with expiration dates of 50%, 100%, and 150% of time to expiration of the American-style option.

B. Testcases 1, 2, 3, and 4

Panel A of Table II presents the approximation results for the different methods for Testcase 1, as the number of simulation paths and the degree of the polynomials in the set of basis functions changes, while Panel B augments the basis functions by including nine additional functions, which correspond to prices of European options with different strikes and maturities. The asymptotic value of the option is reported in the second column of the table. It is computed using 100,000 paths and OLS regression for each corresponding set of basis functions. The standard errors reported throughout the table are computed by running each method 20 times using independent samples. To ease comprehension, we indicate estimates that are at least three standard errors away from the asymptotic value by "***", and estimates that are between two and three standard errors away from the asymptotic value by "**".

From Panel A of Table II we notice that when the number of paths is very small, e.g., 100 or 1,000, OLS regression does poorly. When the number of paths is 100, Quantile regression, MPP, and MMPP outperform the other methods, while when the number of paths is 1,000, Tikhonov regularization, MPP and MMPP outperform. As the number of paths increases to 5,000 and 10,000 all methods other than Quantile regression improve. Adding European options as basis functions in Table II, Panel B, improves the performance of Tikhonov regularization, MPP, and MMPP methods, but deteriorates the performance of OLS regression. Overall, we note that as the number of basis functions increases, the performance of OLS regression deteriorates, while the performance of the Tikhonov regularization, MPP, and MMPP methods is only marginally affected.

The results for the case of the call option on a stock that pays continuous dividends, presented in Table III, are similar.

Table IV corresponds to an American-Asian call. From Panel A we notice that OLS regression outperforms the other methods, other than quantile regression. When European options are added to the basis, in Panel B, Tikhonov regularization and the MMPP method improve to the level of OLS regression.

In Table V we present the results for the case of an American put on an asset that follows jump-diffusion. With a polynomial basis, OLS regression and the MMPP method outperform the rest. When European options are added to the basis in Panel B, the performance of OLS regression deteriorates, while Tikhonov regularization improves to the level of the MMPP method, which is not affected by the change in the basis functions.

C. Testcase 5

Table VI presents the results for the case of a max-call option on two assets for different strike prices. From Panel A, we notice that no method is able to accurately approximate the asymptotic price for the in-the-money option when the basis functions include only polynomials. Panels C and E correspond to at-the-money and out-of-the-money options with polynomial basis functions, and we notice that OLS regression, MPP and Tikhonov regularization are able to better approximate the option prices. Adding European option prices to the basis – Panels B, D, F – improves the performance of Tikhonov regularization, the MPP, and MMPP methods and deteriorates the performance of OLS regression. Quantile regression and CART perform poorly throughout.

Table VII presents the case of a max-call option on ten assets.[7] Almost all methods perform poorly. MMPP does relatively better but still does not consistently achieve accurate estimates.[8]

D. Overall Evaluation of the Five Testcases

Tables VIII and IX summarize the results for all the estimation methods and all the testcases. Each estimation method is given a grade of "+", "–" , or "o", depending on whether it performed well, poorly, or average. Below each method the four columns correspond to 100, 1,000, 5,000, and 10,000 simulation paths. Table VIII summarizes the results for all the sets of basis functions, while Table IX focuses on the case of the sets of basis functions with the largest number of functions.

Overall we find that for a small number of paths OLS regression performs poorly. The MMPP method consistently outperforms OLS regression in the case of 100 and 1,000 paths. As the number of paths increases, OLS regression tends to catch up to the other methods. This is not a surprise since the price estimated using OLS regression converges to the asymptotic price in the limit of a large number of simulation paths.

The regression tree method performs poorly for most of the specifications. Checking the approximate continuation value function generated by this method shows that the size of the tree turns out to be too small to capture the structure of the continuation value function.

For Testcases 1, 2, 4, and 5 with two assets, the quantile regression method converges to a price significantly below the asymptotic price. This suggests a systematic bias that may be caused by the convexity of the continuation value that leads to positive skewness of the distribution of option prices.

[7]Results for the cases of three and five assets are similar and are available from the authors.

[8]We know of no easy way to compute European option prices for this case, and for this reason we have not included European option prices to the basis.

Adding European option prices as basis functions improves the performance of the Tikhonov, MPP, and MMPP methods, but deteriorates the performance of OLS regression. This improvement indicates that methods which can incorporate better information faster, like MPP and MMPP which use a hierarchical structure for choosing the approximation to the continuation value, benefit more from adding the European option prices. Adding basis functions exacerbates the overfitting problem of OLS regression for a small number of simulation paths.

There is no method that consistently provides accurate results for all the testcases with different numbers of basis functions and paths. However, the MMPP method appears to be the most accurate in a consistent basis, especially when functions similar to the option price are available.

E. Performance as the Number of Exercise Dates Increases

Our results so far indicate that OLS regression underperforms the MMPP method in the case of polynomial bases, and the Tikhonov regularization, MPP, and MMPP methods for the case of bases that include both polynomials and European option prices. A potential reason for underperformance is the interaction between the OLS regression estimates of the coefficients of the basis functions and the recursive nature of the estimation, due to the backward induction methodology of optimal stopping problems. We investigate this interaction in Table X, where we compute the prices for Testcase 2, and Testcase 5 with two assets, for varying numbers of exercise dates.

Similar to Tables VIII, and IX, each method is given a grade of "+", "–" , or "o", depending on whether it performed well, poorly, or average. Below each method the four columns correspond to 100, 1,000, 5,000, and 10,000 simulation paths. The results are presented in summary form for a few representative cases as the number of exercise dates changes, for the cases of polynomial bases and bases that include polynomials as well as European option prices.[9] From the table we notice that OLS regression performs better when the number of exercise opportunities is smaller, and that the performance of OLS regression deteriorates faster than the performance of alternatives such as Tikhonov regularization, the MPP and MMPP methods as the number of exercise dates increases. This behavior can be understood by the nature of the methods: since Tikhonov regularization, MPP and MMPP methods penalize the size of the coefficients of the basis functions, the estimates of the coefficients obtained from these methods are likely to be smaller and less variable than those obtained by OLS regression. For the same reason, their out-of-sample performance is likely to be close to in-sample performance compared to OLS regression. Given that the estimates of the coefficients are used in a recursive manner, we conjecture that the more variable coefficient estimates of OLS regression result in the faster propagation of errors compared to the other methods.

V. Conclusions

We investigated the performance of OLS regression in Monte Carlo simulation methods of pricing American options. We found that OLS regression is prone to overfitting and producing inaccurate estimates when the number of simulation paths is small, when the number of functions used to approximate the continuation value function is large, when European option prices are included in the basis functions, and when the number of exercise dates increases. In the case of polynomial bases, an alternative that performs as well as OLS regression and often better is the MMPP method. When European option prices are added to the polynomial bases, Tikhonov regularization, the MPP and MMPP methods outperform OLS regression. Given the increased use of Monte Carlo simulation methods for pricing American options, and the fact that it is often difficult to check whether the number of paths used is sufficiently large for the option being priced,

[9]Additional details are available from the authors.

and the set of basis functions used, we recommend the use of the MMPP method as a way to check the accuracy of the OLS regression estimates.

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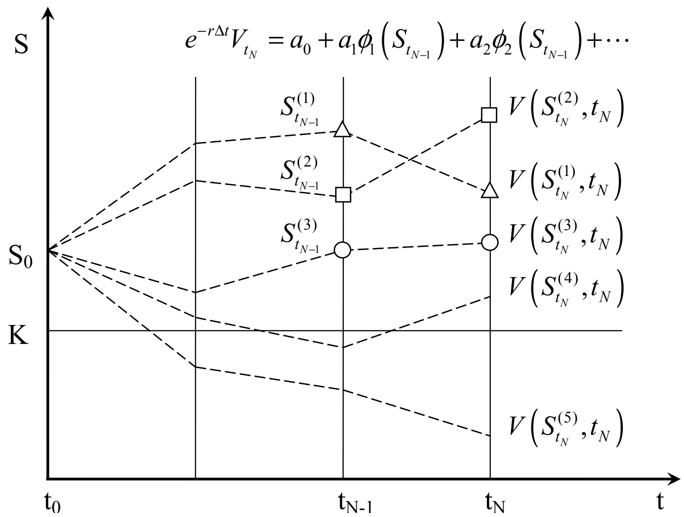

Figure 1. The Least Squares Monte Carlo (LSM) algorithm

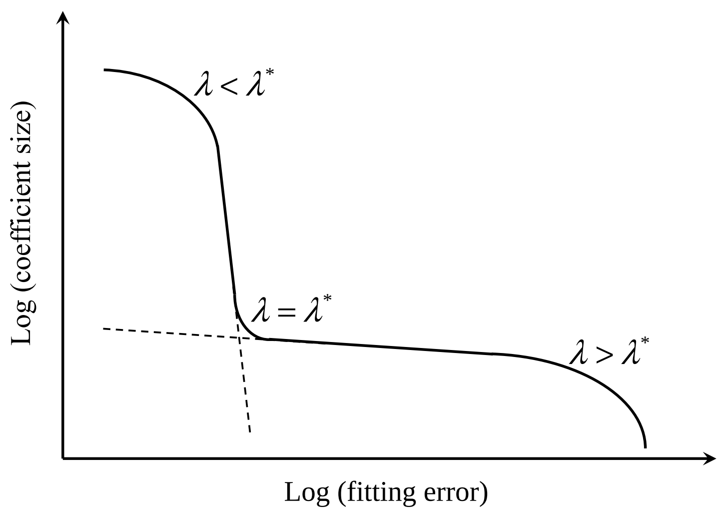

Figure 2. The generic form of an L-curve

Table I
Basis Functions of Max-Call Option On Three Assets

N_b	Basis Functions		
0	1	$S_{(3)}$	$S_{(2)}$
1	S_1	S_2	S_3
2	S_1^2	S_2^2	S_3^2
3	S_1S_2	S_1S_3	S_2S_3
4	S_1^3	S_2^3	S_3^3
5	$S_1^2S_2$	$S_1^2S_3$	$S_2^2S_1$
	$S_2^2S_3$	$S_3^2S_1$	$S_3^2S_2$
	$S_1S_2S_3$		

Notes: $S_1,\ S_2,\ S_3$ are the prices of the three assets. $S_{(3)}$ is the highest price among the three assets, and $S_{(2)}$ is the second highest price. For simplicity we tabulate the basis functions in an incremental style; i.e., if "N_b" equals 2, the basis functions will include not only the three terms $S_1^2,\ S_2^2$, and S_3^2, but also the six terms with "N_b" less than 2.

Table II
Call Option with Discrete Dividends

			Panel A. Polynomial basis				
N_b	**Asymptotic**	**OLS**	**Quantile**	**Tikhonov**	**MPP**	**MMPP**	**CART**
			Number of Paths = 100				
0	7.82±0.00	8.26±0.24	7.72±0.17	7.53± 0.15	7.98±0.18	8.20±0.16**	10.08±0.23***
1	7.80±0.01	7.94±0.20	7.32±0.20**	7.76± 0.18	8.05±0.17	7.89±0.22	
2	7.87±0.01	8.57±0.20***	7.66±0.28	7.85± 0.20	7.53±0.17	8.21±0.19	
3	7.88±0.00	8.64±0.21***	7.62±0.18	8.54± 0.18***	8.36±0.15***	7.96±0.19	
4	7.88±0.01	9.08±0.29***	7.84±0.22	8.65± 0.25***	8.38±0.19**	7.90±0.15	
5	7.88±0.01	9.67±0.21***	7.79±0.20	8.38± 0.13***	8.15±0.23	8.43±0.18***	
6	7.89±0.00	9.66±0.24***	8.38±0.27	8.47± 0.17***	8.42±0.24**	8.46±0.22**	
7	7.88±0.00	9.99±0.29***	7.97±0.18	8.50± 0.23**	8.42±0.18**	8.46±0.29	
8	7.89±0.01	10.44±0.23***	8.51±0.27**	8.56±0.11***	8.39±0.20**	8.61±0.17***	
9	7.88±0.01	10.30±0.19***	8.35±0.23**	8.69±0.20***	8.35±0.26	8.53±0.24**	
10	7.88±0.01	10.07±0.20***	8.32±0.19**	8.79±0.19***	8.53±0.15***	8.44±0.22**	
			Number of Paths = 1000				
0	7.82±0.00	7.84±0.05	7.51±0.08***	7.50± 0.06***	7.80±0.06	7.96±0.08	8.65±0.07***
1	7.80±0.01	7.83±0.05	6.67±0.07***	7.83± 0.05	7.86±0.07	7.91±0.05**	
2	7.87±0.01	7.93±0.05	7.04±0.10***	7.77± 0.06	7.88±0.06	7.89±0.06	
3	7.88±0.00	7.90±0.06	7.07±0.08***	7.91± 0.04	7.89±0.06	7.91±0.06	
4	7.88±0.01	8.00±0.06	7.00±0.08***	7.91± 0.05	7.82±0.04	7.90±0.03	
5	7.88±0.01	8.01±0.06**	7.11±0.07***	7.84±0.05	7.85±0.05	7.94±0.07	
6	7.89±0.00	8.08±0.05***	7.18±0.07***	7.82±0.05	8.08±0.06***	7.87±0.05	
7	7.88±0.00	8.07±0.08**	7.08±0.07***	7.90±0.06	7.94±0.08	7.90±0.05	
8	7.89±0.01	8.26±0.07***	7.27±0.08***	7.93±0.05	7.88±0.06	7.91±0.05	
9	7.88±0.01	8.19±0.05***	7.19±0.07***	7.88±0.07	7.73±0.07**	7.91±0.05	
10	7.88±0.01	8.14±0.06***	7.12±0.06***	7.96±0.06	7.91±0.06	7.95±0.05	
			Number of Paths = 5000				
0	7.82±0.00	7.80±0.02	7.58±0.04***	7.51± 0.02***	7.77±0.02**	7.81±0.02	7.94±0.03
1	7.80±0.01	7.77±0.03	6.60±0.05***	7.83± 0.02	7.82±0.02	7.79±0.03	
2	7.87±0.01	7.87±0.02	7.03±0.04***	7.85± 0.02	7.89±0.03	7.87±0.03	
3	7.88±0.00	7.88±0.02	7.10±0.04***	7.88± 0.02	7.85±0.02	7.83±0.03	
4	7.88±0.01	7.88±0.02	7.27±0.05***	7.88± 0.02	7.84±0.02	7.84±0.02	
5	7.88±0.01	7.91±0.03	7.16±0.06***	7.85± 0.02	7.89±0.02	7.86±0.03	
6	7.89±0.00	7.91±0.03	7.26±0.07***	7.85± 0.03	7.82±0.03**	7.87±0.02	
7	7.88±0.00	7.95±0.03**	7.14±0.05***	7.88±0.03	7.88±0.03	7.88±0.02	
8	7.89±0.01	7.88±0.03	7.18±0.05***	7.90± 0.02	7.81±0.03**	7.90±0.02	
9	7.88±0.01	7.91±0.03	7.06±0.07***	7.90± 0.02	7.84±0.02	7.93±0.02**	
10	7.88±0.01	7.94±0.03	7.24±0.05***	7.83±0.03	7.84±0.02	7.90±0.02	
			Number of Paths = 10000				
0	7.82±0.00	7.85±0.01**	7.54±0.03***	7.49±0.02***	7.83±0.01	7.82±0.02	7.83±0.02**
1	7.80±0.01	7.82±0.02	6.52±0.03***	7.82± 0.02	7.84±0.02	7.85±0.02**	
2	7.87±0.01	7.85±0.02	7.02±0.03***	7.85± 0.01	7.86±0.02	7.84±0.02	
3	7.88±0.00	7.88±0.02	7.14±0.02***	7.84± 0.02	7.86±0.02	7.88±0.02	
4	7.88±0.01	7.87±0.01	7.04±0.06***	7.89± 0.02	7.87±0.02	7.86±0.02	
5	7.88±0.01	7.88±0.02	6.98±0.06***	7.87± 0.02	7.88±0.02	7.88±0.02	
6	7.89±0.00	7.90±0.02	7.20±0.04***	7.85± 0.02	7.87±0.02	7.87±0.02	
7	7.88±0.00	7.89±0.01	7.23±0.03***	7.86± 0.02	7.86±0.01	7.87±0.02	
8	7.89±0.01	7.90±0.01	7.26±0.03***	7.85± 0.02	7.84±0.02**	7.87±0.02	
9	7.88±0.01	7.89±0.02	7.18±0.05***	7.85± 0.02	7.87±0.02	7.87±0.01	
10	7.88±0.01	7.89±0.01	7.18±0.04***	7.87±0.02	7.88±0.01	7.91±0.01**	

Panel B. Polynomial and European price basis							
N_b	**Asymptotic**	**OLS**	**Quantile**	**Tikhonov**	**MPP**	**MMPP**	**CART**
				Number of Paths = 100			
0	7.89±0.01	9.76±0.20***	8.86±0.25***	8.22±0.19	8.13±0.21	8.53±0.21***	9.67±0.14***
1	7.88±0.00	9.88±0.16***	8.78±0.20***	8.50±0.17***	8.46±0.23**	8.33±0.19**	
2	7.88±0.01	10.69±0.21***	9.33±0.21***	8.56±0.19***	8.71±0.20***	8.90±0.25***	
3	7.89±0.00	10.69±0.19***	9.49±0.30***	8.57±0.22***	8.52±0.15***	8.75±0.26***	
4	7.88±0.00	10.54±0.16***	9.59±0.24***	8.78±0.22***	8.64±0.22***	8.18±0.17	
5	7.88±0.01	10.87±0.24***	9.71±0.29***	8.48±0.16***	8.22±0.14**	8.55±0.29**	
6	7.88±0.00	10.65±0.17***	9.85±0.18***	8.83±0.27***	8.36±0.22**	8.31±0.18**	
7	7.89±0.00	10.68±0.30***	10.14±0.22***	8.62±0.23***	8.72±0.21***	8.85±0.23***	
8	7.88±0.01	11.01±0.23***	9.93±0.18***	8.90±0.23***	8.31±0.15**	8.75±0.22***	
9	7.88±0.00	10.66±0.20***	9.62±0.27***	9.02±0.21***	8.47±0.21**	8.25±0.15**	
10	7.88±0.00	10.77±0.25***	10.07±0.16***	8.71 ±0.18***	8.64±0.25***	8.42±0.18**	
				Number of Paths = 1000			
0	7.89±0.01	8.35±0.07***	7.43±0.08***	7.88±0.06	7.82±0.05	8.01±0.06	8.65±0.07***
1	7.88±0.00	8.18±0.09***	7.49±0.10***	7.96±0.06	7.95±0.06	7.93±0.07	
2	7.88±0.01	8.30±0.05***	7.43±0.08***	7.98±0.06	7.98±0.05	7.95±0.07	
3	7.89±0.00	8.24±0.07***	7.39±0.05***	7.93±0.04	7.99±0.05	7.92±0.04	
4	7.88±0.00	8.31±0.04***	7.44±0.09***	7.94±0.05	7.94±0.08	7.80±0.06	
5	7.88±0.01	8.28±0.06***	7.35±0.09***	7.88±0.05	7.86±0.05	8.02±0.06**	
6	7.88±0.00	8.33±0.06***	7.53±0.08***	7.95±0.06	7.99±0.05**	8.00±0.06	
7	7.89±0.00	8.37±0.06***	7.50±0.05***	8.00±0.06	7.93±0.05	7.89±0.06	
8	7.88±0.01	8.37±0.06***	7.65±0.06***	7.98±0.05	7.81±0.05	7.89±0.08	
9	7.88±0.00	8.49±0.05***	7.48±0.07***	7.90±0.08	7.95±0.05	7.89±0.06	
10	7.88±0.00	8.40±0.06***	7.73±0.09	8.01±0.03***	7.91±0.08	8.08±0.05***	
				Number of Paths = 5000			
0	7.89±0.01	7.93±0.02	7.32±0.04***	7.88± 0.02	7.86±0.03	7.91±0.02	7.91±0.04
1	7.88±0.00	7.93±0.03	7.28±0.06***	7.90± 0.01	7.87±0.02	7.93±0.03	
2	7.88±0.01	7.96±0.02***	7.39±0.04***	7.86±0.03	7.84±0.02	7.92±0.02	
3	7.89±0.00	7.86±0.02	7.32±0.05***	7.91± 0.03	7.84±0.02**	7.87±0.03	
4	7.88±0.00	7.94±0.03	7.33±0.07***	7.90± 0.03	7.87±0.03	7.86±0.03	
5	7.88±0.01	7.96±0.02***	7.29±0.06***	7.86±0.03	7.84±0.03	7.90±0.03	
6	7.88±0.00	8.00±0.02***	7.31±0.08***	7.91±0.02	7.87±0.02	7.90±0.03	
7	7.89±0.00	7.94±0.02**	7.43±0.05***	7.86±0.03	7.90±0.02	7.90±0.02	
8	7.88±0.01	7.95±0.03**	7.47±0.07***	7.91±0.02	7.86±0.02	7.91±0.03	
9	7.88±0.00	7.96±0.02***	7.50±0.04***	7.92±0.02	7.88±0.02	7.88±0.02	
10	7.88±0.00	7.95±0.02***	7.55±0.07***	7.90±0.02	7.91±0.03	7.88±0.03	
				Number of Paths = 10000			
0	7.89±0.01	7.89±0.01	7.28±0.04***	7.90± 0.02	7.90±0.02	7.88±0.02	7.81±0.02***
1	7.88±0.00	7.92±0.02	7.38±0.04***	7.89± 0.02	7.88±0.02	7.88±0.01	
2	7.88±0.01	7.84±0.02	7.27±0.04***	7.87± 0.02	7.88±0.02	7.89±0.01	
3	7.89±0.00	7.91±0.02	7.40±0.06***	7.88± 0.02	7.91±0.02	7.89±0.01	
4	7.88±0.00	7.92±0.01***	7.41±0.04***	7.89±0.01	7.88±0.02	7.90±0.02	
5	7.88±0.01	7.90±0.02	7.24±0.06***	7.85± 0.02	7.88±0.01	7.86±0.02	
6	7.88±0.00	7.90±0.02	7.40±0.06***	7.87± 0.02	7.88±0.01	7.86±0.02	
7	7.89±0.00	7.89±0.02	7.41±0.05***	7.85± 0.02	7.88±0.02	7.91±0.02	
8	7.88±0.01	7.89±0.02	7.47±0.04***	7.90± 0.01	7.88±0.02	7.91±0.02	
9	7.88±0.00	7.93±0.02**	7.45±0.06***	7.89±0.01	7.89±0.02	7.89±0.02	
10	7.88±0.00	7.90±0.02	7.42±0.06***	7.88±0.02	7.89±0.01	7.86±0.02	

Notes: The parameters are: $\sigma = 0.2, r = 0.05, T = 3, D = 5.0, S_0 = 100, K = 100$, and exercise points $t \in \{0, 0.5, 1, 1.5, 2, 2.5, 3\}$. N_b denotes the highest degree of the polynomial basis functions. The standard errors are reported after the ± sign based on 20 replications. The asymptotic value is computed by the OLS regression method with 100,000 paths. The "**" sign means two to three standard errors away from the asymptotic value. The "***" sign means more than three standard errors away.

Table III
Call Option with Continuous Dividends

Panel A. Polynomial basis							
N_b	**Asymptotic**	**OLS**	**Quantile**	**Tikhonov**	**MPP**	**MMPP**	**CART**
Number of Paths = 100							
0	7.77±0.01	7.75±0.22	7.45±0.23	7.42± 0.14**	7.99±0.21	7.94±0.20	9.08±0.21***
1	7.81±0.01	8.46±0.22**	7.22±0.26**	7.72±0.19	8.34±0.27	7.40±0.21	
2	7.86±0.00	8.75±0.26***	7.51±0.22	7.89± 0.21	7.82±0.20	8.41±0.27**	
3	7.86±0.01	9.04±0.21***	7.56±0.21	8.17± 0.24	8.18±0.41	8.56±0.22***	
4	7.88±0.01	9.23±0.20***	7.84±0.23	8.62± 0.29**	8.23±0.19	8.55±0.28**	
5	7.87±0.01	9.06±0.20***	8.18±0.31	8.36± 0.31	8.47±0.25**	8.60±0.26**	
6	7.86±0.01	9.68±0.19***	7.91±0.21	8.19± 0.25	8.59±0.13***	8.05±0.23	
7	7.88±0.01	9.41±0.24***	8.56±0.17***	8.59±0.23***	8.09±0.33	8.70±0.22***	
8	7.87±0.01	9.39±0.26***	8.90±0.26***	8.95±0.27***	8.16±0.23	8.69±0.29**	
9	7.87±0.01	10.37±0.31***	8.57±0.20***	8.74±0.19***	8.59±0.26**	8.27±0.28	
10	7.86±0.01	9.98±0.21***	8.44±0.23**	8.62±0.24***	7.95±0.21	8.28±0.22	
Number of Paths = 1000							
0	7.77±0.01	7.69±0.06	7.62±0.09	7.33± 0.06***	7.82±0.08	7.86±0.07	8.59±0.07***
1	7.81±0.01	7.80±0.07	7.02±0.10***	7.74± 0.04	7.83±0.07	7.97±0.06**	
2	7.86±0.00	8.00±0.05**	7.03±0.10***	7.88±0.07	7.83±0.07	7.95±0.07	
3	7.86±0.01	7.86±0.08	7.14±0.12***	7.87± 0.06	7.83±0.07	7.91±0.07	
4	7.88±0.01	7.77±0.08	7.15±0.12***	7.87± 0.08	7.99±0.06	7.85±0.06	
5	7.87±0.01	8.00±0.06**	7.09±0.09***	7.89±0.05	7.85±0.06	7.87±0.07	
6	7.86±0.01	8.03±0.08**	7.22±0.10***	7.91±0.05	7.87±0.06	7.93±0.05	
7	7.88±0.01	8.02±0.07	7.31±0.10***	8.02± 0.08	7.87±0.08	7.86±0.06	
8	7.87±0.01	8.13±0.07***	7.06±0.09***	7.87±0.04	7.90±0.08	7.96±0.07	
9	7.87±0.01	8.10±0.08**	7.33±0.05***	7.90±0.07	7.80±0.06	7.78±0.06	
10	7.86±0.01	8.10±0.07***	7.18±0.10***	7.86±0.08	7.93±0.08	7.91±0.05	
Number of Paths = 5000							
0	7.77±0.01	7.77±0.03	7.66±0.03***	7.40± 0.02***	7.74±0.04	7.77±0.03	7.90±0.03
1	7.81±0.01	7.85±0.04	6.84±0.05***	7.79± 0.03	7.73±0.02***	7.85±0.03	
2	7.86±0.00	7.83±0.03	7.23±0.04***	7.78± 0.03**	7.89±0.03	7.84±0.03	
3	7.86±0.01	7.92±0.03	7.24±0.03***	7.92± 0.02**	7.85±0.02	7.86±0.02	
4	7.88±0.01	7.89±0.03	7.30±0.05***	7.84± 0.02	7.88±0.03	7.86±0.03	
5	7.87±0.01	7.94±0.02***	7.35±0.04***	7.84±0.03	7.85±0.03	7.85±0.02	
6	7.86±0.01	7.89±0.03	7.30±0.05***	7.84± 0.03	7.84±0.03	7.86±0.03	
7	7.88±0.01	7.88±0.03	7.35±0.06***	7.83± 0.03	7.83±0.03	7.85±0.04	
8	7.87±0.01	7.87±0.03	7.30±0.04***	7.91± 0.03	7.85±0.02	7.86±0.02	
9	7.87±0.01	7.90±0.02	7.28±0.06***	7.85± 0.03	7.83±0.03	7.84±0.02	
10	7.86±0.01	7.87±0.03	7.29±0.07***	7.90±0.02	7.83±0.03	7.85±0.03	
Number of Paths = 10000							
0	7.77±0.01	7.76±0.02	7.67±0.03***	7.44± 0.02***	7.78±0.02	7.76±0.02	7.79±0.02***
1	7.81±0.01	7.84±0.02	6.90±0.06***	7.79± 0.02	7.79±0.02	7.84±0.02	
2	7.86±0.00	7.84±0.02	7.14±0.03***	7.77± 0.02***	7.85±0.02	7.83±0.02	
3	7.86±0.01	7.89±0.02	7.22±0.03***	7.84± 0.02	7.87±0.02	7.86±0.02	
4	7.88±0.01	7.89±0.02	7.26±0.04***	7.82± 0.02**	7.89±0.01	7.82±0.01***	
5	7.87±0.01	7.85±0.02	7.35±0.04***	7.85± 0.02	7.87±0.02	7.84±0.03	
6	7.86±0.01	7.86±0.02	7.30±0.03***	7.85± 0.02	7.83±0.02	7.84±0.02	
7	7.88±0.01	7.87±0.02	7.30±0.03***	7.83± 0.02**	7.81±0.03**	7.83±0.02**	
8	7.87±0.01	7.89±0.02	7.37±0.04***	7.85± 0.02	7.84±0.02	7.87±0.02	
9	7.87±0.01	7.93±0.02**	7.38±0.03***	7.86±0.02	7.87±0.02	7.87±0.03	
10	7.86±0.01	7.89±0.02	7.32±0.03***	7.88±0.02	7.80±0.02**	7.84±0.02	

Panel B. Polynomial and European price basis							
N_b	**Asymptotic**	**OLS**	**Quantile**	**Tikhonov**	**MPP**	**MMPP**	**CART**
			Number of Paths = 100				
0	7.87±0.01	7.57±0.19	7.73±0.21	7.61± 0.22	7.79±0.19	7.84±0.23	8.77±0.20***
1	7.86±0.01	7.81±0.23	7.44±0.21	7.99± 0.25	7.99±0.25	7.83±0.25	
2	7.86±0.01	8.74±0.31**	7.19±0.23**	8.05±0.20	8.26±0.20	8.44±0.25**	
3	7.86±0.00	9.33±0.23***	7.44±0.26	8.78± 0.25***	8.24±0.22	8.49±0.24**	
4	7.86±0.01	9.16±0.27***	7.86±0.19	8.66± 0.26***	8.23±0.18**	8.15±0.25	
5	7.87±0.01	9.75±0.25***	8.24±0.31	8.76± 0.29***	8.17±0.30	8.06±0.22	
6	7.87±0.01	9.51±0.19***	8.53±0.26**	8.64±0.31**	8.58±0.21***	8.32±0.18**	
7	7.87±0.01	9.12±0.21***	8.32±0.20**	8.63±0.24***	8.58±0.18***	7.99±0.24	
8	7.87±0.01	10.05±0.32***	8.60±0.20***	8.90±0.25***	8.08±0.20	8.73±0.21***	
9	7.86±0.01	9.87±0.20***	8.59±0.24***	8.56±0.31**	8.23±0.19	8.37±0.26	
10	7.86±0.01	9.80±0.23***	8.90±0.27***	8.63±0.19***	8.11±0.20	8.25±0.29	
			Number of Paths = 1000				
0	7.87±0.01	7.76±0.07	7.66±0.08**	7.45± 0.05***	7.74±0.06**	7.91±0.07	8.63±0.05***
1	7.86±0.01	7.98±0.07	6.89±0.11***	7.85± 0.06	7.93±0.08	7.98±0.06	
2	7.86±0.01	7.88±0.06	7.02±0.08***	7.79± 0.08	7.86±0.07	7.97±0.08	
3	7.86±0.00	7.93±0.06	7.14±0.11***	7.90± 0.06	7.83±0.05	7.80±0.06	
4	7.86±0.01	8.02±0.07**	7.25±0.11***	7.92±0.09	7.85±0.05	7.84±0.06	
5	7.87±0.01	8.02±0.05**	7.36±0.09***	7.81±0.05	7.81±0.06	7.85±0.05	
6	7.87±0.01	8.05±0.08**	7.08±0.10***	7.84±0.07	7.93±0.06	7.92±0.06	
7	7.87±0.01	8.01±0.08	7.17±0.09***	7.95± 0.08	7.79±0.03**	7.93±0.07	
8	7.87±0.01	8.09±0.07***	7.10±0.09***	7.88±0.08	7.88±0.05	7.94±0.07	
9	7.86±0.01	8.04±0.06**	7.31±0.11***	7.94±0.07	7.81±0.05	7.83±0.08	
10	7.86±0.01	8.16±0.06***	7.18±0.08***	7.82±0.05	7.87±0.05	7.86±0.06	
			Number of Paths = 5000				
0	7.87±0.01	7.78±0.03**	7.73±0.05**	7.48±0.02***	7.81±0.03	7.80±0.04	7.91±0.03
1	7.86±0.01	7.83±0.02	6.89±0.07***	7.79± 0.03**	7.77±0.03**	7.80±0.03	
2	7.86±0.01	7.88±0.03	7.22±0.04***	7.82± 0.03	7.84±0.03	7.91±0.04	
3	7.86±0.00	7.90±0.03	7.20±0.06***	7.86± 0.03	7.86±0.04	7.82±0.03	
4	7.86±0.01	7.87±0.03	7.26±0.05***	7.86± 0.03	7.84±0.04	7.84±0.03	
5	7.87±0.01	7.90±0.02	7.34±0.05***	7.80± 0.03**	7.89±0.04	7.89±0.02	
6	7.87±0.01	7.91±0.02	7.32±0.05***	7.90± 0.03	7.82±0.02**	7.84±0.04	
7	7.87±0.01	7.91±0.02	7.32±0.04***	7.85± 0.03	7.81±0.03	7.90±0.03	
8	7.87±0.01	7.90±0.02	7.40±0.06***	7.88± 0.03	7.81±0.03	7.85±0.03	
9	7.86±0.01	7.91±0.03	7.32±0.05***	7.88± 0.03	7.83±0.03	7.84±0.03	
10	7.86±0.01	7.88±0.03	7.35±0.05***	7.87±0.03	7.84±0.03	7.85±0.03	
			Number of Paths = 10000				
0	7.87±0.01	7.79±0.02***	7.61±0.04***	7.43±0.02***	7.74±0.02***	7.74±0.02***	7.82±0.02
1	7.86±0.01	7.78±0.02***	6.90±0.05***	7.75±0.03***	7.85±0.02	7.82±0.02	
2	7.86±0.01	7.88±0.01	7.18±0.03***	7.76± 0.02***	7.84±0.02	7.83±0.03	
3	7.86±0.00	7.87±0.02	7.26±0.03***	7.81± 0.02**	7.86±0.02	7.84±0.02	
4	7.86±0.01	7.89±0.02	7.28±0.04***	7.84± 0.02	7.84±0.02	7.86±0.02	
5	7.87±0.01	7.87±0.02	7.27±0.03***	7.85± 0.01	7.84±0.02	7.82±0.02**	
6	7.87±0.01	7.84±0.01**	7.30±0.04***	7.85±0.02	7.84±0.02	7.86±0.02	
7	7.87±0.01	7.85±0.01	7.29±0.05***	7.83± 0.02	7.87±0.02	7.85±0.02	
8	7.87±0.01	7.91±0.02	7.31±0.04***	7.83± 0.02	7.82±0.02**	7.84±0.02	
9	7.86±0.01	7.89±0.02	7.24±0.04***	7.88± 0.02	7.85±0.02	7.86±0.02	
10	7.86±0.01	7.89±0.02	7.23±0.05***	7.85±0.01	7.85±0.02	7.86±0.02	

Notes: The parameters are: $\sigma = 0.2, r = 0.05, T = 3, \delta = 0.1, S_0 = 100, K = 100$, and exercise points $t \in \{0, 0.5, 1, 1.5, 2, 2.5, 3\}$. N_b denotes the highest degree of the polynomial basis functions. The standard errors are reported after the ± sign based on 20 replications. The asymptotic value is computed by the OLS regression method with 100,000 paths. The "**" sign means two to three standard errors away from the asymptotic value. The "***" sign means more than three standard errors away.

Table IV
American-Asian Call Option

N_b	Asymptotic	OLS	Quantile	Tikhonov	MPP	MMPP	CART
Panel A. Polynomial basis							
Number of Paths = 100							
0	5.42±0.00	5.38±0.10	5.44±0.11	5.44± 0.08	5.39±0.11	5.32±0.11	5.39±0.10
1	5.52±0.00	5.27±0.10**	5.59±0.11	5.60± 0.10	5.39±0.09	5.36±0.12	
2	5.52±0.00	5.75±0.11**	5.57±0.11	5.47± 0.12	5.56±0.10	5.57±0.09	
3	5.52±0.00	5.52±0.09	5.65±0.13	5.29± 0.10**	5.19±0.08***	5.33±0.11	
4	5.52±0.00	5.52±0.13	5.59±0.12	5.52± 0.13	5.56±0.10	5.40±0.11	
5	5.52±0.00	5.41±0.10	5.40±0.13	5.39± 0.12	5.36±0.11	5.31±0.1**	
6	5.52±0.00	5.56±0.11	5.34±0.11	5.49± 0.08	5.31±0.07**	5.42±0.12	
7	5.52±0.00	5.56±0.11	5.36±0.10	5.31± 0.08**	5.34±0.08**	5.41±0.12	
8	5.52±0.00	5.38±0.13	5.53±0.12	5.42± 0.11	5.30±0.10**	5.26±0.09**	
9	5.52±0.00	5.78±0.11**	5.54±0.11	5.54± 0.11	5.49±0.08	5.35±0.08**	
10	5.52±0.00	5.66±0.08	5.54±0.11	5.72± 0.14	5.33±0.09**	5.39±0.09	
Number of Paths = 1000							
0	5.42±0.00	5.42±0.03	5.33±0.03**	5.41± 0.04	5.42±0.04	5.41±0.04	5.43±0.03**
1	5.52±0.00	5.47±0.04	5.53±0.04	5.38± 0.03***	5.39±0.03***	5.53±0.03	
2	5.52±0.00	5.54±0.04	5.49±0.03	5.41± 0.04**	5.48±0.04	5.42±0.04**	
3	5.52±0.00	5.51±0.03	5.49±0.03	5.43± 0.03**	5.39±0.03***	5.45±0.05	
4	5.52±0.00	5.56±0.04	5.53±0.04	5.42± 0.04**	5.42±0.04**	5.44±0.04	
5	5.52±0.00	5.55±0.03	5.53±0.03	5.56± 0.04	5.38±0.04***	5.46±0.03	
6	5.52±0.00	5.59±0.03**	5.47±0.04	5.43± 0.03**	5.39±0.03***	5.44±0.04	
7	5.52±0.00	5.57±0.03	5.49±0.03	5.49± 0.04	5.39±0.04***	5.38±0.03***	
8	5.52±0.00	5.52±0.03	5.47±0.02**	5.46± 0.04	5.43±0.03**	5.41±0.03***	
9	5.52±0.00	5.50±0.04	5.48±0.03	5.51± 0.04	5.45±0.04	5.41±0.03***	
10	5.52±0.00	5.54±0.03	5.53±0.03	5.49± 0.04	5.45±0.03**	5.47±0.04	
Number of Paths = 5000							
0	5.42±0.00	5.43±0.02	5.43±0.01	5.39± 0.02	5.42±0.01	5.43±0.02	5.44±0.02***
1	5.52±0.00	5.51±0.02	5.52±0.01	5.42± 0.01***	5.45±0.01***	5.53±0.01	
2	5.52±0.00	5.57±0.01***	5.52±0.01	5.40± 0.01***	5.39±0.01***	5.42±0.01***	
3	5.52±0.00	5.52±0.02	5.52±0.01	5.39± 0.02***	5.43±0.01***	5.41±0.02***	
4	5.52±0.00	5.52±0.02	5.51±0.02	5.39± 0.01***	5.41±0.02***	5.45±0.01***	
5	5.52±0.00	5.50±0.01	5.51±0.02	5.50± 0.02	5.41±0.01***	5.46±0.02**	
6	5.52±0.00	5.50±0.02	5.51±0.02	5.49± 0.02	5.42±0.01***	5.49±0.02	
7	5.52±0.00	5.53±0.01	5.50±0.02	5.48± 0.02	5.41±0.01***	5.47±0.01***	
8	5.52±0.00	5.51±0.01	5.54±0.02	5.45± 0.02***	5.40±0.02***	5.41±0.02***	
9	5.52±0.00	5.52±0.02	5.51±0.02	5.45± 0.01***	5.39±0.01***	5.42±0.01***	
10	5.52±0.00	5.53±0.01	5.52±0.02	5.48± 0.02	5.40±0.01***	5.43±0.01***	
Number of Paths = 10000							
0	5.42±0.00	5.41±0.01	5.41±0.01	5.39± 0.01**	5.42±0.01	5.43±0.01	5.48±0.01***
1	5.52±0.00	5.53±0.01	5.52±0.01	5.40± 0.01***	5.47±0.01***	5.53±0.01	
2	5.52±0.00	5.54±0.01	5.51±0.01	5.39± 0.01***	5.42±0.01***	5.43±0.01***	
3	5.52±0.00	5.53±0.01	5.51±0.01	5.38± 0.01***	5.43±0.01***	5.44±0.01***	
4	5.52±0.00	5.51±0.01	5.52±0.01	5.42± 0.01***	5.43±0.01***	5.51±0.01	
5	5.52±0.00	5.52±0.01	5.50±0.01	5.48± 0.01***	5.43±0.01***	5.50±0.01	
6	5.52±0.00	5.55±0.01**	5.54±0.01	5.49± 0.01**	5.38±0.01***	5.49±0.01**	
7	5.52±0.00	5.52±0.01	5.52±0.01	5.47± 0.01***	5.40±0.01***	5.45±0.01***	
8	5.52±0.00	5.53±0.01	5.53±0.01	5.49± 0.01**	5.40±0.01***	5.44±0.01***	
9	5.52±0.00	5.51±0.01	5.52±0.01	5.47± 0.01***	5.43±0.01***	5.46±0.01***	
10	5.52±0.00	5.52±0.01	5.51±0.01	5.49± 0.01**	5.42±0.01***	5.43±0.01***	

Panel B. Polynomial and European price basis							
N_b	**Asymptotic**	**OLS**	**Quantile**	**Tikhonov**	**MPP**	**MMPP**	**CART**
			Number of Paths = 100				
0	5.78±0.00	5.84±0.07	5.81±0.10	5.89± 0.09	5.74±0.12	5.73±0.12	5.50±0.11**
1	5.78±0.00	5.73±0.12	5.93±0.09	5.84± 0.12	5.77±0.10	5.68±0.11	
2	5.78±0.00	5.71±0.10	5.78±0.12	5.80± 0.09	5.66±0.11	5.72±0.10	
3	5.78±0.00	5.73±0.09	5.75±0.11	5.81± 0.08	5.58±0.10	5.96±0.10	
4	5.78±0.00	5.88±0.14	5.68±0.14	5.79± 0.08	5.74±0.09	5.89±0.12	
5	5.78±0.00	5.84±0.08	5.97±0.13	5.75± 0.08	5.84±0.12	5.91±0.15	
6	5.78±0.00	5.78±0.10	5.63±0.13	5.94± 0.14	5.76±0.08	5.88±0.16	
7	5.79±0.00	5.63±0.10	5.80±0.10	5.76± 0.11	5.72±0.10	5.56±0.09**	
8	5.78±0.00	5.86±0.09	5.91±0.11	5.82± 0.10	5.81±0.09	5.76±0.11	
9	5.79±0.00	5.88±0.11	5.76±0.13	5.60± 0.10	5.70±0.12	5.76±0.13	
10	5.78±0.00	5.83±0.09	5.90±0.09	5.75± 0.08	5.75±0.09	5.69±0.11	
			Number of Paths = 1000				
0	5.78±0.00	5.76±0.03	5.78±0.03	5.81± 0.02	5.81±0.03	5.76±0.03	5.48±0.03***
1	5.78±0.00	5.82±0.04	5.77±0.03	5.74± 0.03	5.80±0.04	5.77±0.03	
2	5.78±0.00	5.78±0.04	5.69±0.04**	5.74± 0.03	5.81±0.03	5.76±0.03	
3	5.78±0.00	5.79±0.03	5.80±0.04	5.72± 0.03	5.75±0.03	5.78±0.02	
4	5.78±0.00	5.82±0.03	5.73±0.03	5.80± 0.03	5.82±0.04	5.74±0.04	
5	5.78±0.00	5.78±0.04	5.77±0.03	5.79± 0.03	5.76±0.04	5.77±0.03	
6	5.78±0.00	5.81±0.04	5.73±0.03	5.77± 0.04	5.82±0.04	5.76±0.03	
7	5.79±0.00	5.70±0.02***	5.82±0.04	5.78± 0.03	5.72±0.03**	5.81±0.03	
8	5.78±0.00	5.79±0.03	5.81±0.03	5.81± 0.04	5.79±0.03	5.77±0.03	
9	5.79±0.00	5.80±0.03	5.83±0.03	5.77± 0.04	5.76±0.04	5.78±0.03	
10	5.78±0.00	5.81±0.03	5.86±0.04	5.81± 0.03	5.77±0.04	5.74±0.03	
			Number of Paths = 5000				
0	5.78±0.00	5.78±0.02	5.79±0.02	5.80± 0.02	5.78±0.01	5.78±0.02	5.44±0.02***
1	5.78±0.00	5.80±0.02	5.78±0.02	5.79± 0.02	5.79±0.02	5.81±0.02	
2	5.78±0.00	5.78±0.01	5.76±0.01	5.78± 0.01	5.76±0.02	5.77±0.02	
3	5.78±0.00	5.77±0.02	5.79±0.02	5.80± 0.02	5.77±0.01	5.78±0.01	
4	5.78±0.00	5.78±0.02	5.76±0.01	5.77± 0.01	5.78±0.02	5.80±0.01	
5	5.78±0.00	5.76±0.01	5.76±0.01	5.80± 0.01	5.80±0.02	5.81±0.01**	
6	5.78±0.00	5.77±0.01	5.75±0.01**	5.78± 0.02	5.79±0.02	5.75±0.02	
7	5.79±0.00	5.80±0.02	5.78±0.01	5.78± 0.02	5.81±0.01	5.79±0.02	
8	5.78±0.00	5.80±0.02	5.81±0.02	5.79± 0.02	5.77±0.02	5.77±0.02	
9	5.79±0.00	5.77±0.01	5.77±0.01	5.76± 0.02	5.79±0.01	5.77±0.01	
10	5.78±0.00	5.77±0.01	5.74±0.01***	5.77±0.02	5.78±0.01	5.77±0.02	
			Number of Paths = 10000				
0	5.78±0.00	5.80±0.01	5.75±0.01**	5.81± 0.01**	5.79±0.01	5.79±0.01	5.45±0.01***
1	5.78±0.00	5.78±0.01	5.78±0.01	5.79± 0.01	5.78±0.01	5.77±0.01	
2	5.78±0.00	5.77±0.01	5.76±0.01	5.77± 0.01	5.78±0.01	5.78±0.01	
3	5.78±0.00	5.79±0.01	5.77±0.01	5.78± 0.01	5.78±0.01	5.78±0.01	
4	5.78±0.00	5.78±0.01	5.76±0.01	5.79± 0.01	5.78±0.01	5.80±0.01	
5	5.78±0.00	5.76±0.01	5.78±0.01	5.78± 0.01	5.79±0.01	5.78±0.01	
6	5.78±0.00	5.78±0.01	5.78±0.01	5.81± 0.01**	5.78±0.01	5.78±0.01	
7	5.79±0.00	5.77±0.01	5.77±0.01	5.78± 0.01	5.77±0.01	5.81±0.01	
8	5.78±0.00	5.79±0.01	5.77±0.01	5.79± 0.01	5.79±0.01	5.79±0.01	
9	5.79±0.00	5.79±0.01	5.78±0.01	5.77± 0.01	5.78±0.01	5.78±0.01	
10	5.78±0.00	5.78±0.01	5.78±0.01	5.77± 0.01	5.79±0.01	5.77±0.01	

Notes: The parameters are: $\sigma = 0.2, r = 0.09, T = 120/365, t' = 91/365, S_0 = 100, K = 100$, and exercise points $t \in \{0, 105/365, 108/365, 111/365, 114/365, 117/365, 120/365\}$. N_b denotes the highest degree of the polynomial basis functions. The standard errors are reported after the ± sign based on 20 replications. The asymptotic value is computed by the OLS regression method with 100,000 paths. The "**" sign means two to three standard errors away from the asymptotic value. The "***" sign means more than three standard errors away.

Table V
Put Option on a Jump-Diffusion Asset

			Panel A. Polynomial basis				
N_b	**Asymptotic**	**OLS**	**Quantile**	**Tikhonov**	**MPP**	**MMPP**	**CART**
			Number of Paths = 100				
0	8.65±0.01	8.77±0.21	9.02±0.30	8.25± 0.29	9.12±0.32	8.90±0.28	10.03±0.21***
1	8.68±0.01	9.20±0.26	8.12±0.29	8.30± 0.19	9.06±0.21	9.20±0.27	
2	8.71±0.01	8.93±0.32	8.56±0.24	8.88± 0.25	8.95±0.21	8.87±0.26	
3	8.73±0.01	9.09±0.11***	8.56±0.30	8.80± 0.23	8.82±0.21	8.16±0.22**	
4	8.72±0.01	8.95±0.18	8.55±0.29	9.36± 0.17***	9.58±0.30**	9.13±0.27	
5	8.72±0.01	9.59±0.23***	8.48±0.21	8.82± 0.23	8.93±0.29	8.87±0.28	
6	8.71±0.01	9.56±0.20***	9.05±0.30	9.31± 0.31	8.58±0.27	8.76±0.33	
7	8.72±0.01	9.13±0.24	8.99±0.25	8.94± 0.27	8.57±0.21	8.59±0.30	
8	8.72±0.01	9.59±0.24***	9.23±0.21**	8.83±0.28	9.08±0.27	8.77±0.31	
9	8.73±0.01	9.38±0.24**	9.09±0.32	9.76± 0.25***	8.80±0.22	9.17±0.32	
10	8.73±0.01	9.81±0.26***	9.06±0.26	8.91±0.28	8.60±0.33	8.72±0.32	
			Number of Paths = 1000				
0	8.65±0.01	8.66±0.08	8.88±0.08**	8.11± 0.09***	8.64±0.08	8.60±0.10	9.12±0.07***
1	8.68±0.01	8.76±0.10	8.29±0.14**	8.51± 0.08**	8.59±0.08	8.59±0.10	
2	8.71±0.01	8.80±0.08	8.33±0.10***	8.46± 0.08***	8.65±0.07	8.55±0.08	
3	8.73±0.01	8.83±0.08	8.39±0.12**	8.81± 0.11	8.52±0.09**	8.70±0.08	
4	8.72±0.01	8.73±0.07	8.37±0.11***	8.64± 0.08	8.66±0.08	8.65±0.08	
5	8.72±0.01	8.74±0.07	8.35±0.11***	8.75± 0.08	8.63±0.09	8.74±0.07	
6	8.71±0.01	8.85±0.09	8.33±0.11***	8.56± 0.08	8.63±0.06	8.64±0.10	
7	8.72±0.01	8.83±0.10	8.20±0.14***	8.75± 0.07	8.50±0.10**	8.77±0.10	
8	8.72±0.01	8.81±0.09	8.19±0.09***	8.77± 0.09	8.42±0.12**	8.81±0.07	
9	8.73±0.01	8.92±0.08**	8.29±0.09***	8.51±0.07***	8.47±0.09**	8.86±0.08	
10	8.73±0.01	8.78±0.07	8.26±0.10***	8.72±0.08	8.63±0.12	8.62±0.08	
			Number of Paths = 5000				
0	8.65±0.01	8.70±0.04	8.62±0.03	8.06± 0.04***	8.65±0.03	8.64±0.03	8.73±0.04
1	8.68±0.01	8.73±0.04	8.40±0.06***	8.57± 0.03***	8.68±0.04	8.73±0.04	
2	8.71±0.01	8.69±0.04	8.32±0.03***	8.64± 0.03**	8.68±0.03	8.59±0.03***	
3	8.73±0.01	8.79±0.05	8.35±0.05***	8.66± 0.04	8.60±0.04***	8.66±0.04	
4	8.72±0.01	8.72±0.03	8.36±0.05***	8.68± 0.04	8.47±0.03***	8.67±0.03	
5	8.72±0.01	8.73±0.03	8.35±0.05***	8.69± 0.04	8.46±0.04***	8.73±0.03	
6	8.71±0.01	8.78±0.04	8.31±0.09***	8.75± 0.04	8.54±0.04***	8.72±0.04	
7	8.72±0.01	8.76±0.03	8.34±0.06***	8.71± 0.04	8.64±0.03**	8.73±0.03	
8	8.72±0.01	8.76±0.04	8.21±0.08***	8.75± 0.03	8.61±0.04**	8.74±0.03	
9	8.73±0.01	8.76±0.03	8.22±0.09***	8.69± 0.04	8.66±0.04	8.69±0.04	
10	8.73±0.01	8.77±0.03	8.34±0.09***	8.68±0.03	8.68±0.03	8.69±0.05	
			Number of Paths = 10000				
0	8.65±0.01	8.65±0.03	8.74±0.03**	8.06± 0.02***	8.62±0.03	8.67±0.02	8.65±0.02***
1	8.68±0.01	8.64±0.02	8.32±0.03***	8.56± 0.01***	8.73±0.02**	8.68±0.03	
2	8.71±0.01	8.76±0.02**	8.34±0.03***	8.52±0.03***	8.64±0.03**	8.68±0.03	
3	8.73±0.01	8.77±0.03	8.39±0.04***	8.72± 0.02	8.60±0.03***	8.70±0.02	
4	8.72±0.01	8.76±0.03	8.32±0.03***	8.63± 0.03**	8.48±0.02***	8.72±0.03	
5	8.72±0.01	8.72±0.03	8.15±0.06***	8.69± 0.03	8.60±0.03***	8.71±0.03	
6	8.71±0.01	8.75±0.03	7.95±0.11***	8.73± 0.01	8.60±0.03***	8.70±0.02	
7	8.72±0.01	8.71±0.02	8.33±0.06***	8.68± 0.03	8.67±0.03	8.71±0.02	
8	8.72±0.01	8.71±0.01	8.31±0.04***	8.69± 0.03	8.62±0.02***	8.68±0.03	
9	8.73±0.01	8.71±0.02	8.25±0.07***	8.67± 0.03	8.63±0.02***	8.64±0.02***	
10	8.73±0.01	8.78±0.02**	8.23±0.06***	8.67±0.03	8.65±0.03**	8.65±0.03**	

Panel B. Polynomial and European price basis

N_b	Asymptotic	OLS	Quantile	Tikhonov	MPP	MMPP	CART
			Number of Paths = 100				
0	8.72±0.01	9.74±0.25***	9.60±0.29***	9.10±0.34	9.49±0.17***	9.10±0.27	9.55±0.24***
1	8.72±0.01	9.94±0.23***	9.62±0.27***	8.71±0.27	8.84±0.23	9.05±0.26	
2	8.72±0.01	10.32±0.24***	9.41±0.30**	8.92±0.25	9.03±0.24	8.90±0.28	
3	8.73±0.01	9.55±0.19***	9.34±0.31	9.09± 0.23	9.07±0.28	9.15±0.31	
4	8.72±0.01	9.29±0.23**	9.84±0.21***	8.90±0.17	8.90±0.25	9.66±0.22***	
5	8.72±0.01	10.03±0.32***	9.56±0.28**	9.61±0.30**	9.35±0.23**	9.02±0.29	
6	8.71±0.01	9.77±0.26***	9.80±0.31***	9.21±0.18**	8.66±0.23	8.83±0.27	
7	8.72±0.01	10.17±0.21***	9.73±0.32***	9.13±0.21	9.17±0.24	8.74±0.33	
8	8.73±0.01	9.36±0.29**	9.65±0.24***	8.58±0.18	9.09±0.26	9.44±0.24**	
9	8.72±0.01	10.14±0.24***	9.90±0.28***	9.28±0.20**	9.47±0.28**	8.91±0.22	
10	8.73±0.01	9.54±0.31**	9.49±0.29**	8.78±0.30	9.33±0.21**	8.89±0.28	
			Number of Paths = 1000				
0	8.72±0.01	8.83±0.09	8.44±0.08***	8.81± 0.07	8.72±0.08	8.73±0.07	9.12±0.10***
1	8.72±0.01	8.91±0.07**	8.31±0.11***	8.70±0.08	8.65±0.08	8.64±0.08	
2	8.72±0.01	8.92±0.10	8.59±0.10	8.75± 0.07	8.79±0.08	8.69±0.08	
3	8.73±0.01	8.82±0.09	8.49±0.10**	8.81± 0.08	8.61±0.08	8.78±0.08	
4	8.72±0.01	8.94±0.07***	8.54±0.12	8.79± 0.07	8.80±0.10	8.63±0.06	
5	8.72±0.01	8.97±0.07***	8.53±0.08**	8.72±0.06	8.58±0.06**	8.92±0.07**	
6	8.71±0.01	8.82±0.09	8.68±0.09	8.77± 0.06	8.72±0.09	8.79±0.10	
7	8.72±0.01	9.01±0.11**	8.58±0.12	8.77± 0.08	8.63±0.07	8.60±0.10	
8	8.73±0.01	8.94±0.08**	8.57±0.10	8.75± 0.08	8.61±0.07	8.69±0.10	
9	8.72±0.01	8.92±0.08**	8.70±0.11	8.82± 0.07	8.60±0.08	8.78±0.09	
10	8.73±0.01	9.09±0.06***	8.75±0.09	8.83±0.09	8.96±0.08**	8.64±0.07	
			Number of Paths = 5000				
0	8.72±0.01	8.73±0.04	8.41±0.05***	8.68± 0.04	8.69±0.04	8.71±0.03	8.77±0.04
1	8.72±0.01	8.74±0.03	8.39±0.05***	8.72± 0.03	8.64±0.03**	8.72±0.04	
2	8.72±0.01	8.68±0.04	8.49±0.04***	8.71± 0.03	8.65±0.04	8.69±0.03	
3	8.73±0.01	8.76±0.04	8.42±0.06***	8.73± 0.04	8.62±0.03***	8.72±0.04	
4	8.72±0.01	8.68±0.03	8.38±0.08***	8.71± 0.04	8.70±0.04	8.75±0.04	
5	8.72±0.01	8.74±0.04	8.46±0.06***	8.74± 0.03	8.68±0.03	8.66±0.02**	
6	8.71±0.01	8.82±0.04**	8.45±0.05***	8.78±0.02***	8.64±0.03**	8.64±0.05	
7	8.72±0.01	8.74±0.04	8.51±0.05***	8.78± 0.03	8.69±0.05	8.70±0.03	
8	8.73±0.01	8.77±0.03	8.49±0.05***	8.71± 0.03	8.75±0.03	8.69±0.03	
9	8.72±0.01	8.76±0.03	8.45±0.06***	8.74± 0.03	8.71±0.04	8.79±0.03**	
10	8.73±0.01	8.78±0.03	8.46±0.05***	8.73±0.03	8.67±0.03	8.79±0.04	
			Number of Paths = 10000				
0	8.72±0.01	8.74±0.03	8.33±0.04***	8.71± 0.03	8.69±0.02	8.67±0.02**	8.66±0.02***
1	8.72±0.01	8.75±0.02	8.39±0.04***	8.75± 0.03	8.71±0.03	8.68±0.02	
2	8.72±0.01	8.74±0.03	8.42±0.06***	8.75± 0.02	8.71±0.03	8.74±0.02	
3	8.73±0.01	8.78±0.02**	8.50±0.04***	8.71±0.03	8.66±0.02***	8.73±0.02	
4	8.72±0.01	8.77±0.03	8.42±0.06***	8.73± 0.03	8.67±0.02**	8.76±0.02	
5	8.72±0.01	8.75±0.03	8.35±0.04***	8.77± 0.03	8.74±0.03	8.71±0.02	
6	8.71±0.01	8.74±0.02	8.43±0.04***	8.71± 0.02	8.75±0.02	8.70±0.03	
7	8.72±0.01	8.71±0.03	8.48±0.05***	8.69± 0.02	8.65±0.03**	8.71±0.02	
8	8.73±0.01	8.74±0.02	8.49±0.04***	8.75± 0.03	8.73±0.02	8.75±0.02	
9	8.72±0.01	8.71±0.02	8.47±0.05***	8.71± 0.02	8.68±0.02	8.69±0.03	
10	8.73±0.01	8.75±0.03	8.47±0.04***	8.75±0.02	8.71±0.02	8.73±0.03	

Notes: The parameters are: $\sigma = \sqrt{0.08}, r = 0.1, T = 0.5, \delta = 0.2, S_0 = 100, K = 100, \lambda = 2$, and exercise points $t \in \{0, 0.125, 0.25, 0.375, 0.5\}$. N_b denotes the highest degree of the polynomial basis functions. The standard errors are reported after the ± sign based on 20 replications. The asymptotic value is computed by the OLS regression method with 100,000 paths. The "**" sign means two to three standard errors away from the asymptotic value. The "***" sign means more than three standard errors away.

Table VI
Max-Call Option on Two Assets

N$_b$	Asymptotic	OLS	Quantile	Tikhonov	MPP	MMPP	CART
Panel A. K=80, polynomial basis							
N$_b$	**Asymptotic**	**OLS**	**Quantile**	**Tikhonov**	**MPP**	**MMPP**	**CART**
Number of Paths = 100							
0	24.82±0.00	25.81±0.18***	25.16±0.15**	23.67 ±0.21***	23.87±0.23***	25.20±0.21	25.54±0.17***
1	24.82±0.01	26.08±0.22***	25.71±0.25***	25.97±0.18***	23.59±0.16***	25.73±0.20***	
2	24.87±0.00	26.86±0.17***	25.50±0.22**	26.41 ±0.12***	24.77±0.23	25.75±0.20***	
3	24.98±0.01	26.49±0.15***	25.90±0.21***	25.99±0.19***	24.91±0.23	26.10±0.19***	
4	25.00±0.01	27.35±0.19***	26.28±0.23***	26.51±0.20***	25.33±0.18	25.57±0.16***	
5	25.00±0.00	27.88±0.21***	25.94±0.20***	26.78±0.18***	25.21±0.19	25.74±0.13***	
Number of Paths = 1000							
0	24.82±0.00	24.88±0.05	24.77±0.05	23.74±0.06***	23.78±0.06***	24.84±0.06	23.87±0.06***
1	24.82±0.01	25.04±0.04***	24.83±0.05	24.91±0.05	23.83±0.05***	24.96±0.06**	
2	24.87±0.00	25.30±0.05***	24.95±0.06	25.12±0.05***	24.73±0.05**	25.00±0.05**	
3	24.98±0.01	25.17±0.06***	25.08±0.06	25.07±0.06	24.59±0.07***	24.99±0.05	
4	25.00±0.01	25.31±0.07***	25.18±0.05***	25.23±0.05***	24.80±0.06***	24.81±0.05***	
5	25.00±0.00	25.45±0.06***	25.07±0.06	25.03±0.04	24.80±0.05***	24.82±0.07**	
Number of Paths = 5000							
0	24.82±0.00	24.82±0.03	24.67±0.03***	23.71±0.02***	24.00±0.03***	24.85±0.03	23.71±0.02***
1	24.82±0.01	24.87±0.02**	24.73±0.02***	24.84 ±0.02	24.03±0.03***	24.90±0.03**	
2	24.87±0.00	25.01±0.03***	24.73±0.03***	24.96±0.03**	24.79±0.03**	24.88±0.02	
3	24.98±0.01	25.06±0.02***	24.90±0.02***	24.97±0.03	24.76±0.02***	24.92±0.03	
4	25.00±0.01	25.05±0.02**	24.88±0.02***	24.97 ±0.02	24.80±0.03***	24.88±0.02***	
5	25.00±0.00	25.14±0.03***	24.94±0.02**	24.98 ±0.03	24.80±0.03***	24.91±0.03**	
Number of Paths = 10000							
0	24.82±0.00	24.84±0.02	24.65±0.02***	23.69±0.02***	24.27±0.02***	24.84±0.02	23.70±0.02***
1	24.82±0.01	24.85±0.02	24.67±0.02***	24.86±0.02	24.23±0.01***	24.82±0.02	
2	24.87±0.00	24.93±0.02**	24.70±0.01***	24.92 ±0.02**	24.85±0.01	24.91±0.02	
3	24.98±0.01	25.00±0.02	24.82±0.02***	24.94±0.02	24.81±0.02***	24.91±0.02***	
4	25.00±0.01	25.04±0.01***	24.88±0.02***	24.95±0.02**	24.80±0.02***	24.88±0.02***	
5	25.00±0.00	25.04±0.02	24.84±0.01***	24.93±0.02***	24.73±0.02***	24.93±0.02***	
Panel B. K=80, polynomial and European price basis							
N$_b$	**Asymptotic**	**OLS**	**Quantile**	**Tikhonov**	**MPP**	**MMPP**	**CART**
Number of Paths = 100							
0	25.03±0.00	26.88±0.15***	26.03±0.18***	25.26±0.18	25.69±0.16***	26.02±0.19***	25.62±0.21**
1	25.03±0.01	27.76±0.17***	26.94±0.27***	25.83±0.19***	26.04±0.23***	25.81±0.11***	
2	25.04±0.01	28.08±0.19***	26.89±0.17***	26.19±0.18***	25.73±0.17***	26.06±0.16***	
3	25.03±0.00	27.82±0.16***	27.13±0.16***	26.22±0.19***	25.69±0.23**	26.33±0.20***	
4	25.03±0.01	28.47±0.20***	27.35±0.25***	26.16±0.22***	25.80±0.20***	26.17±0.21***	
5	25.04±0.01	28.73±0.21***	27.88±0.19***	26.00±0.16***	25.93±0.27***	26.02±0.18***	
Number of Paths = 1000							
0	25.03±0.00	24.99±0.05	25.38±0.06***	25.09±0.05	25.09±0.05	25.09±0.04	23.60±0.06***
1	25.03±0.01	25.29±0.05***	25.47±0.05***	25.02±0.04	25.05±0.05	25.10±0.04	
2	25.04±0.01	25.18±0.06**	25.50±0.06***	25.19 ±0.05**	25.03±0.05	25.16±0.05**	
3	25.03±0.00	25.33±0.04***	25.55±0.04***	25.09±0.05	25.08±0.05	25.13±0.05	
4	25.03±0.01	25.36±0.06***	25.58±0.06***	25.07±0.06	25.05±0.04	25.21±0.05***	
5	25.04±0.01	25.46±0.08***	25.66±0.05***	25.10±0.05	25.02±0.04	25.20±0.05***	
Number of Paths = 5000							
0	25.03±0.00	24.90±0.03***	25.13±0.03***	25.00±0.02	25.00±0.02	25.05±0.02	23.69±0.03***
1	25.03±0.01	24.90±0.03***	25.12±0.02***	25.03±0.02	25.04±0.02	25.06±0.02	
2	25.04±0.01	24.99±0.03	25.14±0.03***	25.02±0.03	25.02±0.02	25.06±0.02	
3	25.03±0.00	24.97±0.03	25.16±0.02***	25.00±0.02	24.99±0.02	25.05±0.02	
4	25.03±0.01	24.98±0.03	25.14±0.03***	25.04±0.02	24.99±0.02	25.08±0.02**	
5	25.04±0.01	25.01±0.02	25.16±0.02***	25.04±0.02	25.00±0.02	25.04±0.02	
Number of Paths = 10000							
0	25.03±0.00	24.82±0.01***	25.08±0.02**	25.00 ±0.02	25.03±0.01	25.02±0.02	23.68±0.02***
1	25.03±0.01	24.85±0.02***	25.08±0.01***	25.01±0.02	25.00±0.01**	25.03±0.01	
2	25.04±0.01	24.85±0.02***	25.10±0.02**	25.03 ±0.02	25.00±0.02	25.02±0.02	
3	25.03±0.00	24.89±0.02***	25.10±0.01***	25.01±0.02	25.04±0.01	25.04±0.02	
4	25.03±0.01	24.89±0.02***	25.10±0.02***	25.05±0.01	25.03±0.01	25.05±0.01	
5	25.04±0.01	24.89±0.02***	25.09±0.01***	25.04±0.01	25.03±0.02	25.05±0.01	

Panel C. K=100, polynomial basis							
N_b	**Asymptotic**	**OLS**	**Quantile**	**Tikhonov**	**MPP**	**MMPP**	**CART**
				Number of Paths = 100			
0	9.43±0.01	10.01±0.22**	8.99±0.23	8.48± 0.15***	8.97±0.16**	9.20±0.19	9.98±0.15**
1	9.43±0.00	10.79±0.19***	9.25±0.19	9.63±0.22	8.98±0.16**	9.45±0.18	
2	9.51±0.00	11.27±0.22***	10.23±0.17***	10.33 ±0.18***	9.21±0.18	9.58±0.20	
3	9.52±0.01	11.41±0.20***	9.98±0.29	10.76±0.11***	9.49±0.12	9.73±0.16	
4	9.52±0.01	11.57±0.14***	9.79±0.24	11.24±0.19***	9.16±0.15**	9.85±0.23	
5	9.53±0.01	11.58±0.21***	9.65±0.16	10.93±0.21***	9.87±0.25	10.07±0.16***	
				Number of Paths = 1000			
0	9.43±0.01	9.48±0.05	9.22±0.07**	8.75± 0.05***	8.82±0.06***	9.37±0.07	8.86±0.06***
1	9.43±0.00	9.59±0.08	9.13±0.07***	9.53± 0.04**	8.89±0.05***	9.39±0.08	
2	9.51±0.00	9.70±0.06***	9.33±0.08**	9.67±0.08	9.12±0.06***	9.58±0.06	
3	9.52±0.01	9.80±0.07***	9.37±0.08	9.72± 0.06***	9.16±0.04***	9.55±0.07	
4	9.52±0.01	9.86±0.06***	9.41±0.07	9.70± 0.06**	9.25±0.04***	9.54±0.07	
5	9.53±0.01	9.91±0.06***	9.43±0.06	9.68± 0.05**	9.27±0.05***	9.51±0.05	
				Number of Paths = 5000			
0	9.43±0.01	9.44±0.03	9.20±0.02***	8.73± 0.02***	8.96±0.03***	9.47±0.03	8.72±0.02***
1	9.43±0.00	9.49±0.02**	9.23±0.03***	9.40±0.04	8.99±0.03***	9.42±0.03	
2	9.51±0.00	9.56±0.02**	9.37±0.02***	9.53±0.03	9.26±0.02***	9.50±0.03	
3	9.52±0.01	9.58±0.03	9.39±0.03***	9.57± 0.03	9.27±0.02***	9.51±0.03	
4	9.52±0.01	9.58±0.02**	9.39±0.03***	9.56±0.02	9.34±0.03***	9.49±0.03	
5	9.53±0.01	9.59±0.03	9.37±0.03***	9.55± 0.02	9.31±0.01***	9.53±0.02	
				Number of Paths = 10000			
0	9.43±0.01	9.45±0.02	9.24±0.02***	8.72± 0.01***	9.04±0.02***	9.43±0.02	8.78±0.02***
1	9.43±0.00	9.44±0.02	9.21±0.02***	9.43± 0.02	9.05±0.02***	9.43±0.02	
2	9.51±0.00	9.52±0.02	9.37±0.02***	9.55± 0.02	9.31±0.02***	9.53±0.02	
3	9.52±0.01	9.54±0.02	9.33±0.02***	9.50± 0.02	9.31±0.01***	9.52±0.02	
4	9.52±0.01	9.60±0.02***	9.38±0.02***	9.53±0.02	9.32±0.03***	9.49±0.02	
5	9.53±0.01	9.54±0.02	9.34±0.03***	9.56± 0.01**	9.38±0.02***	9.52±0.02	

Panel D. K=100, polynomial and European price basis							
N_b	**Asymptotic**	**OLS**	**Quantile**	**Tikhonov**	**MPP**	**MMPP**	**CART**
				Number of Paths = 100			
0	9.55±0.01	11.64±0.18***	10.18±0.22**	10.10±0.23**	10.26±0.16***	10.34±0.23***	9.74±0.20
1	9.57±0.01	11.84±0.18***	10.65±0.18***	10.82 ±0.19***	10.14±0.21**	10.67±0.15***	
2	9.56±0.01	12.21±0.19***	11.06±0.18***	10.52 ±0.18***	9.97±0.16**	10.50±0.19***	
3	9.57±0.01	12.30±0.18***	11.60±0.17***	10.88 ±0.20***	9.93±0.21	10.49±0.22***	
4	9.57±0.00	12.23±0.21***	11.62±0.16***	10.88 ±0.14***	9.90±0.18	10.16±0.16***	
5	9.56±0.01	12.47±0.20***	11.86±0.26***	10.93 ±0.20***	10.01±0.17**	10.81±0.24***	
				Number of Paths = 1000			
0	9.55±0.01	10.00±0.07***	9.33±0.09**	9.64±0.06	9.59±0.05	9.60±0.04	8.95±0.07***
1	9.57±0.01	9.98±0.06***	9.55±0.05	9.68± 0.07	9.68±0.06	9.61±0.07	
2	9.56±0.01	10.13±0.05***	9.62±0.05	9.66±0.08	9.53±0.05	9.69±0.06**	
3	9.57±0.01	10.12±0.06***	9.68±0.07	9.66±0.07	9.72±0.07**	9.60±0.05	
4	9.57±0.00	10.15±0.06***	9.79±0.06***	9.58±0.06	9.63±0.06	9.61±0.07	
5	9.56±0.01	10.25±0.06***	9.75±0.08**	9.68±0.06	9.50±0.04	9.69±0.06**	
				Number of Paths = 5000			
0	9.55±0.01	9.65±0.02***	9.33±0.04***	9.53±0.03	9.56±0.03	9.60±0.03	8.75±0.02***
1	9.57±0.01	9.67±0.02***	9.37±0.04***	9.62±0.02**	9.55±0.02	9.60±0.03	
2	9.56±0.01	9.70±0.02***	9.38±0.04***	9.59±0.03	9.56±0.02	9.59±0.03	
3	9.57±0.01	9.71±0.02***	9.45±0.03***	9.57±0.03	9.57±0.02	9.57±0.02	
4	9.57±0.00	9.69±0.02***	9.49±0.02***	9.53±0.03	9.55±0.03	9.61±0.03	
5	9.56±0.01	9.74±0.02***	9.45±0.03***	9.59±0.02	9.59±0.03	9.56±0.02	
				Number of Paths = 10000			
0	9.55±0.01	9.63±0.01***	9.32±0.03***	9.51±0.02	9.52±0.01**	9.56±0.02	8.76±0.01***
1	9.57±0.01	9.61±0.02	9.34±0.03***	9.55± 0.02	9.53±0.02	9.53±0.02	
2	9.56±0.01	9.62±0.02**	9.35±0.03***	9.54±0.02	9.53±0.02	9.57±0.01	
3	9.57±0.01	9.62±0.02**	9.35±0.02***	9.53±0.01***	9.56±0.02	9.54±0.02	
4	9.57±0.00	9.66±0.02***	9.43±0.03***	9.56±0.02	9.55±0.02	9.55±0.02	
5	9.56±0.01	9.64±0.02***	9.34±0.03***	9.57±0.02	9.54±0.02	9.56±0.02	

Panel E. K=110, polynomial basis							
N$_b$	**Asymptotic**	**OLS**	**Quantile**	**Tikhonov**	**MPP**	**MMPP**	**CART**
Number of Paths = 100							
0	4.95±0.00	5.81±0.19***	4.37±0.37	4.65± 0.13**	5.12±0.15	4.91±0.18	5.12±0.09
1	Not enough in-the-money paths beyond this point						
Number of Paths = 1000							
0	4.95±0.00	4.98±0.06	4.62±0.07***	4.75± 0.05***	4.73±0.05***	4.81±0.04***	5.05±0.04
1	4.95±0.01	5.12±0.04***	4.64±0.09***	5.06±0.04**	4.81±0.05**	4.98±0.06	
2	4.99±0.01	5.21±0.04***	4.90±0.07	5.18± 0.06***	4.89±0.03***	5.02±0.06	
3	4.99±0.00	5.19±0.06***	4.69±0.06***	5.15±0.05***	4.82±0.04***	5.07±0.05	
4	4.99±0.00	5.18±0.05***	4.77±0.06***	5.15±0.06**	4.96±0.05	5.01±0.05	
5	5.00±0.01	5.28±0.05***	4.85±0.07**	5.18±0.06**	4.86±0.05**	4.95±0.05	
Number of Paths = 5000							
0	4.95±0.00	4.98±0.02	4.64±0.02***	4.74± 0.02***	4.80±0.02***	4.94±0.03	4.71±0.02***
1	4.95±0.01	4.98±0.02	4.74±0.03***	4.98± 0.02	4.78±0.02***	4.93±0.02	
2	4.99±0.01	5.00±0.02	4.84±0.02***	5.03± 0.02	4.87±0.02***	4.97±0.02	
3	4.99±0.00	5.06±0.02***	4.78±0.02***	5.02±0.02	4.89±0.02***	4.98±0.02	
4	4.99±0.00	5.03±0.02	4.79±0.02***	5.05± 0.02**	4.86±0.02***	5.06±0.02***	
5	5.00±0.01	5.07±0.02***	4.77±0.03***	5.06±0.02**	4.86±0.02***	5.03±0.02	
Number of Paths = 10000							
0	4.95±0.00	4.99±0.02	4.73±0.03***	4.70± 0.02***	4.78±0.01***	4.93±0.02	4.74±0.01***
1	4.95±0.01	4.96±0.02	4.72±0.02***	4.95± 0.02	4.82±0.02***	4.95±0.02	
2	4.99±0.01	4.99±0.02	4.80±0.02***	5.02± 0.02	4.88±0.02***	4.97±0.01	
3	4.99±0.00	5.01±0.02	4.79±0.03***	5.02± 0.02	4.86±0.01***	4.99±0.01	
4	4.99±0.00	5.02±0.02	4.80±0.02***	5.03± 0.01***	4.88±0.02***	4.98±0.02	
5	5.00±0.01	5.04±0.01***	4.78±0.03***	4.98±0.02	4.89±0.02***	5.00±0.01	
Panel F. K=110, polynomial and European price basis							
N$_b$	**Asymptotic**	**OLS**	**Quantile**	**Tikhonov**	**MPP**	**MMPP**	**CART**
Number of Paths = 100							
0	Not enough in-the-money paths beyond this point						
Number of Paths = 1000							
0	5.02±0.01	5.35±0.04***	4.84±0.06**	5.03±0.05	5.08±0.04	5.10±0.04	5.06±0.04
1	5.02±0.00	5.45±0.06***	4.95±0.06	5.19± 0.05***	4.99±0.07	5.20±0.06**	
2	5.01±0.00	5.52±0.05***	5.03±0.06	5.12± 0.05**	5.05±0.05	5.20±0.04***	
3	5.02±0.01	5.44±0.06***	5.00±0.05	5.29± 0.06***	5.10±0.05	5.04±0.04	
4	5.02±0.01	5.43±0.05***	5.08±0.06	5.15± 0.07	5.09±0.06	5.16±0.07	
5	5.02±0.01	5.48±0.03***	5.22±0.06***	5.30±0.05***	5.14±0.04**	5.19±0.05***	
Number of Paths = 5000							
0	5.02±0.01	5.08±0.02**	4.77±0.03***	5.02±0.02	5.02±0.02	5.08±0.02**	4.78±0.02***
1	5.02±0.00	5.14±0.02***	4.87±0.03***	5.05±0.02	5.04±0.03	5.05±0.02	
2	5.01±0.00	5.14±0.02***	4.92±0.03**	5.02±0.03	4.99±0.02	5.04±0.02	
3	5.02±0.01	5.16±0.02***	4.87±0.04***	5.02±0.02	5.04±0.02	5.05±0.02	
4	5.02±0.01	5.16±0.03***	4.92±0.03***	5.04±0.02	4.96±0.02**	5.05±0.03	
5	5.02±0.01	5.16±0.03***	4.92±0.03***	5.02±0.02	5.01±0.01	5.06±0.03	
Number of Paths = 10000							
0	5.02±0.01	5.07±0.02**	4.77±0.02***	5.01±0.01	5.03±0.01	5.03±0.01	4.73±0.02***
1	5.02±0.00	5.06±0.01***	4.83±0.02***	5.02±0.02	5.03±0.01	5.03±0.01	
2	5.01±0.00	5.07±0.02**	4.85±0.02***	5.02±0.02	5.01±0.01	5.03±0.01	
3	5.02±0.01	5.07±0.02**	4.85±0.03***	5.02±0.02	5.00±0.02	5.02±0.02	
4	5.02±0.01	5.09±0.02***	4.85±0.03***	5.03±0.01	4.98±0.02	5.02±0.02	
5	5.02±0.01	5.09±0.02***	4.86±0.02***	4.99±0.01**	4.99±0.02	5.01±0.01	

Notes: The parameters are: $\sigma = 0.2, r = 0.05, T = 1, \delta = 0.1, S_0 = 100, \rho = 0.3$, and exercise points $t \in \{0, 1/12, 2/12, \ldots, 12/12\}$. N_b implicitly shows the number of basis functions according to Table I. The standard errors are reported after the ± sign based on 20 replications. The asymptotic value is computed by the OLS regression method with 100,000 paths. The "**" sign means two to three standard errors away from the asymptotic value. The "***" sign means more than three standard errors away.

Table VII
Max-Call Option on Ten Assets

Panel A. K=80, polynomial basis							
N_b	**Asymptotic**	**OLS**	**Quantile**	**Tikhonov**	**MPP**	**MMPP**	**CART**
			Number of Paths = 100				
0	40.38±0.01	41.09±0.24**	40.67±0.25	36.66±0.65***	37.94±0.31***	40.80±0.29	38.00±0.21***
1	40.39±0.01	42.60±0.22***	41.58±0.27***	38.00±0.29***	38.24±0.30***	42.34±0.21***	
2	40.40±0.01	43.17±0.25***	41.58±0.27***	38.79±0.26***	39.24±0.19***	42.28±0.24***	
3	40.42±0.01	45.53±0.23***	42.46±0.20***	39.43±0.22***	39.78±0.25**	41.76±0.26***	
4	40.41±0.00	46.74±0.25***	43.54±0.24***	40.18±0.24	39.76±0.26**	41.92±0.17***	
5	Not enough in-the-money paths beyond this point						
			Number of Paths = 1000				
0	40.38±0.01	40.46±0.07	40.26±0.07	37.32±0.10***	38.54±0.09***	40.44±0.08	37.02±0.07***
1	40.39±0.01	40.69±0.09***	40.84±0.08***	37.77±0.10***	38.83±0.08***	40.60±0.05***	
2	40.40±0.01	40.95±0.08***	40.78±0.09***	38.62±0.08***	39.72±0.09***	40.47±0.06	
3	40.42±0.01	41.59±0.07***	40.87±0.06***	38.86±0.07***	39.35±0.09***	40.62±0.09**	
4	40.41±0.00	41.62±0.06***	40.91±0.06***	39.32±0.08***	39.75±0.09***	40.56±0.08	
5	40.54±0.01	44.46±0.08***	43.04±0.08***	39.54±0.08***	39.66±0.08***	40.49±0.08	
			Number of Paths = 5000				
0	40.38±0.01	40.40±0.03	40.41±0.04	37.25±0.04***	39.97±0.04***	40.42±0.03	36.95±0.05***
1	40.39±0.01	40.44±0.03	40.43±0.03	37.79±0.04***	39.98±0.04***	40.47±0.03**	
2	40.40±0.01	40.46±0.03	40.49±0.03**	38.51±0.04***	40.15±0.04***	40.36±0.03	
3	40.42±0.01	40.71±0.03***	40.59±0.03***	38.89±0.03***	40.03±0.04***	40.40±0.04	
4	40.41±0.00	40.71±0.03***	40.56±0.04***	39.25±0.03***	40.11±0.04***	40.42±0.03	
5	40.54±0.01	41.72±0.03***	41.17±0.03***	39.42±0.03***	40.07±0.03***	40.36±0.03***	
			Number of Paths = 10000				
0	40.38±0.01	40.40±0.03	40.41±0.02	37.28±0.03***	40.21±0.03***	40.38±0.02	36.91±0.03***
1	40.39±0.01	40.45±0.02**	40.45±0.02**	37.77±0.03***	40.22±0.03***	40.43±0.02	
2	40.40±0.01	40.45±0.03	40.49±0.03**	38.50±0.02***	40.17±0.03***	40.37±0.02	
3	40.42±0.01	40.56±0.02***	40.52±0.02***	38.92±0.03***	40.11±0.03***	40.41±0.03	
4	40.41±0.00	40.60±0.03***	40.56±0.02***	39.12±0.03***	40.14±0.02***	40.37±0.02	
5	40.54±0.01	41.19±0.02***	40.95±0.03***	39.41±0.02***	40.16±0.03***	40.43±0.02***	
Panel B. K=100, polynomial basis							
N_b	**Asymptotic**	**OLS**	**Quantile**	**Tikhonov**	**MPP**	**MMPP**	**CART**
			Number of Paths = 100				
0	21.79±0.01	22.35±0.25**	22.39±0.22**	19.11±0.28***	19.98±0.24***	22.00±0.25	19.98±0.19***
1	21.80±0.01	23.68±0.27***	23.03±0.25***	19.03±0.29***	19.61±0.18***	23.46±0.26***	
2	21.81±0.01	24.13±0.20***	22.93±0.30***	19.75±0.19***	20.77±0.23***	23.18±0.22***	
3	21.83±0.01	27.19±0.27***	24.26±0.18***	19.94±0.23***	20.60±0.23***	23.03±0.22***	
4	21.84±0.01	27.89±0.27***	25.97±0.25***	20.49±0.21***	20.86±0.33**	23.62±0.27***	
5	Not enough in-the-money paths beyond this point						
			Number of Paths = 1000				
0	21.79±0.01	21.86±0.08	22.01±0.08**	18.98±0.08***	19.95±0.10***	21.68±0.06	19.00±0.06***
1	21.80±0.01	22.09±0.09***	21.91±0.05**	19.23 ±0.07***	19.80±0.06***	22.10±0.08***	
2	21.81±0.01	22.27±0.07***	22.16±0.06***	19.61±0.10***	20.55±0.08***	21.94±0.06**	
3	21.83±0.01	22.78±0.09***	22.34±0.09***	20.04±0.10***	20.57±0.08***	21.96±0.07	
4	21.84±0.01	22.97±0.06***	22.27±0.07***	20.20±0.06***	20.78±0.09***	21.94±0.08	
5	21.94±0.01	25.50±0.07***	24.14±0.07***	20.59±0.10***	20.75±0.08***	21.96±0.09	
			Number of Paths = 5000				
0	21.79±0.01	21.81±0.03	21.82±0.03	19.01±0.03***	21.15±0.05***	21.83±0.03	18.90±0.04***
1	21.80±0.01	21.85±0.03	21.89±0.03**	19.23±0.03***	21.04±0.03***	21.85±0.03	
2	21.81±0.01	21.94±0.03***	21.91±0.04**	19.65 ±0.02***	21.33±0.04***	21.77±0.03	
3	21.83±0.01	22.16±0.03***	22.00±0.04***	19.96±0.04***	21.03±0.04***	21.85±0.04	
4	21.84±0.01	22.17±0.03***	21.94±0.03***	20.13±0.04***	21.15±0.04***	21.80±0.04	
5	21.94±0.01	23.09±0.03***	22.39±0.04***	20.65±0.03***	21.16±0.04***	21.88±0.04	
			Number of Paths = 10000				
0	21.79±0.01	21.81±0.02	21.82±0.02	18.95±0.02***	21.55±0.02***	21.78±0.02	18.98±0.02***
1	21.80±0.01	21.81±0.02	21.84±0.02	19.20±0.02***	21.48±0.02***	21.87±0.03**	
2	21.81±0.01	21.88±0.02***	21.91±0.03***	19.63±0.02***	21.47±0.02***	21.78±0.03	
3	21.83±0.01	21.95±0.02***	21.90±0.03**	19.91 ±0.02***	21.31±0.02***	21.78±0.03	
4	21.84±0.01	22.00±0.02***	21.93±0.03**	20.15 ±0.02***	21.26±0.02***	21.77±0.02***	
5	21.94±0.01	22.54±0.02***	22.24±0.03***	20.62±0.02***	21.30±0.02***	21.82±0.02***	

Panel C. K=110, polynomial basis							
N$_b$	**Asymptotic**	**OLS**	**Quantile**	**Tikhonov**	**MPP**	**MMPP**	**CART**
Number of Paths = 100							
0	13.85±0.00	13.77±0.24	13.77±0.23	12.54±0.21***	12.38±0.19***	13.17±0.22***	13.46±0.19**
1	13.86±0.01	15.79±0.18***	14.48±0.22**	12.72 ±0.24***	12.48±0.18***	14.26±0.23	
2	Not enough in-the-money paths beyond this point						
Number of Paths = 1000							
0	13.85±0.00	13.94±0.07	13.72±0.08	12.12±0.06***	12.65±0.08***	13.78±0.08	12.36±0.07***
1	13.86±0.01	14.10±0.06***	14.04±0.06**	12.35 ±0.06***	12.57±0.07***	14.06±0.09**	
2	13.88±0.01	14.31±0.06***	14.04±0.10	12.52±0.05***	13.03±0.08***	13.92±0.10	
3	13.89±0.01	14.92±0.07***	13.95±0.07	12.72±0.06***	12.90±0.07***	14.00±0.05**	
4	13.89±0.01	15.10±0.05***	14.19±0.08***	12.72±0.07***	13.12±0.07***	14.07±0.08**	
5	Not enough in-the-money paths beyond this point						
Number of Paths = 5000							
0	13.85±0.00	13.84±0.03	13.84±0.03	12.22±0.03***	13.13±0.04***	13.90±0.03	12.22±0.03***
1	13.86±0.01	13.94±0.03**	13.88±0.02	12.35±0.02***	13.03±0.03***	13.91±0.04	
2	13.88±0.01	14.01±0.03***	13.88±0.03	12.53±0.03***	13.39±0.03***	13.82±0.04	
3	13.89±0.01	14.19±0.03***	13.90±0.03	12.66±0.02***	13.27±0.03***	13.88±0.04	
4	13.89±0.01	14.22±0.03***	13.92±0.03	12.80±0.02***	13.32±0.03***	13.90±0.03	
5	14.00±0.01	14.86±0.02***	14.29±0.03***	13.03±0.03***	13.27±0.03***	13.83±0.03***	
Number of Paths = 10000							
0	13.85±0.00	13.84±0.02	13.87±0.02	12.20±0.03***	13.38±0.02***	13.92±0.02***	12.27±0.02***
1	13.86±0.01	13.85±0.02	13.86±0.02	12.34±0.02***	13.33±0.03***	13.90±0.02	
2	13.88±0.01	13.96±0.02***	13.96±0.03**	12.49 ±0.02***	13.55±0.03***	13.88±0.03	
3	13.89±0.01	14.07±0.03***	13.93±0.02	12.61±0.02***	13.45±0.02***	13.84±0.02**	
4	13.89±0.01	14.06±0.03***	13.95±0.02**	12.79 ±0.02***	13.43±0.02***	13.90±0.02	
5	14.00±0.01	14.49±0.02***	14.09±0.03**	13.00 ±0.03***	13.35±0.03***	13.95±0.03	

Notes: The parameters are: $\sigma = 0.2, r = 0.05, T = 1, \delta = 0.1, S_0 = 100, K = 110, \rho = 0.3$, and exercise points $t \in \{0, 1/12, 2/12, \ldots, 12/12\}$. N_b implicitly shows the number of basis functions according to Table I. The standard errors are reported after the ± sign based on 20 replications. The asymptotic value is computed by the OLS regression method with 100,000 paths. The "**" sign means two to three standard errors away from the asymptotic value. The "***" sign means more than three standard errors away.

Table VIII: Overall Performance Comparison

Case		**OLS**				**Quantile**				**Tikhonov**				**MPP**				**MMPP**				**CART**			
Panel A. Polynomial basis																									
Case 1		−	o	+	+	+	−	−	−	−	+	+	+	o	+	+	+	o	+	+	+	−	−	+	o
Case 2		−	o	+	+	o	−	−	−	o	+	+	+	+	+	+	+	o	+	+	+	−	−	+	−
Case 3		+	+	+	+	+	+	+	+	+	+	o	−	+	o	−	−	+	+	−	o	+	o	−	−
Case 4		o	+	+	+	+	−	−	−	+	+	+	+	+	+	o	−	+	+	+	+	−	−	+	−
Case 5	($n = 2, K = 80$)	−	−	o	+	−	+	−	−	−	o	+	o	o	−	−	−	−	o	o	o	−	−	−	−
Case 5	($n = 2, K = 100$)	−	o	+	+	+	o	−	−	−	o	+	+	+	−	−	−	+	+	+	+	o	−	−	−
Case 5	($n = 2, K = 110$)	−	−	o	+	+	−	−	−	o	−	o	o	+	o	−	−	+	+	+	+	+	+	−	−
Case 5	($n = 3, K = 80$)	−	−	o	o	−	o	+	o	o	−	−	−	o	−	−	−	o	o	o	o	+	−	−	−
Case 5	($n = 3, K = 100$)	−	−	o	o	o	+	o	−	−	−	−	−	o	−	−	−	+	+	o	−	+	−	−	−
Case 5	($n = 3, K = 110$)	−	−	o	+	+	+	−	−	o	−	−	−	−	−	−	−	+	+	o	o	+	−	−	−
Case 5	($n = 5, K = 80$)	−	−	−	o	−	−	o	+	o	−	−	−	o	−	−	−	−	+	o	o	−	−	−	−
Case 5	($n = 5, K = 100$)	−	−	o	o	−	o	+	+	−	−	−	−	o	−	−	−	o	+	+	o	−	−	−	−
Case 5	($n = 5, K = 110$)	−	−	o	o	+	+	+	o	o	−	−	−	−	−	−	−	+	+	+	o	+	−	−	−
Case 5	($n = 10, K = 80$)	−	−	o	o	o	−	o	o	o	−	−	−	o	−	−	−	o	+	+	+	−	−	−	−
Case 5	($n = 10, K = 100$)	−	−	o	o	−	−	o	o	−	−	−	−	−	−	−	−	o	+	+	o	−	−	−	−
Case 5	($n = 10, K = 110$)	o	o	−	o	+	+	+	+	−	−	−	−	−	−	−	−	o	+	+	+	o	−	−	−
Panel B. Polynomial and European price basis																									
Case 1		−	−	o	+	−	−	−	−	−	+	+	+	−	+	+	+	−	+	+	+	−	−	+	−
Case 2		−	o	+	+	o	−	−	−	o	+	+	+	+	+	+	+	+	+	+	+	−	−	+	+
Case 3		+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	o	−	−	−
Case 4		−	o	+	+	−	+	−	−	+	+	+	+	+	+	+	+	+	+	+	+	−	−	+	−
Case 5	($n = 2, K = 80$)	−	−	o	−	−	−	−	−	−	+	+	+	−	+	+	+	−	o	+	+	o	−	−	−
Case 5	($n = 2, K = 100$)	−	−	−	o	−	o	−	−	−	+	+	+	o	+	+	+	−	+	+	+	+	−	−	−
Case 5	($n = 2, K = 110$)		−	−	−		+	−	−		o	+	+		+	+	+		o	+	+		+	−	−

Notes: "+" , "o" , and "–", respectively represent good, neutral, and poor performance. The four columns for each method correspond to the results with different numbers of simulation paths (100, 1,000, 5,000, and 10,000 simulation paths respectively from left to the right). For case 5, n is the number of underlying assets and K is the strike price. Some cells have missing values because of not enough in-the-money paths for the specification.

Table IX: Performance Comparison with the Largest Number of Basis Functions

Case		OLS				Quantile				Tikhonov				MPP				MMPP				CART			
Panel A. Polynomial basis																									
Case 1		−	−	+	+	o	−	−	−	−	+	+	+	−	+	+	+	o	+	+	o	−	−	+	o
Case 2		−	−	+	+	o	−	−	−	−	+	+	+	+	+	+	o	+	+	+	+	−	−	+	−
Case 3		+	+	+	+	+	+	+	+	+	+	+	o	o	o	−	−	+	+	−	−	+	o	−	−
Case 4		−	+	+	o	+	−	−	−	+	+	+	+	+	+	+	o	+	+	+	o	−	−	+	−
Case 5	($n = 2, K = 80$)	−	−	−	+	−	+	o	−	−	+	+	−	+	−	−	−	−	o	o	−	−	−	−	−
Case 5	($n = 2, K = 100$)	−	−	+	+	+	+	−	−	−	o	+	o	+	−	−	−	−	+	+	+	o	−	−	−
Case 5	($n = 2, K = 110$)		−	−	−		o	−	−		o	o	+		o	−	−		+	+	+	+	+	−	−
Case 5	($n = 3, K = 80$)	−	−	−	−	−	−	+	+	+	−	−	−	+	−	−	−	−	o	−	−	+	−	−	−
Case 5	($n = 3, K = 100$)	−	−	−	−	−	o	+	+	+	−	−	−	+	−	−	−	o	o	o	−	+	−	−	−
Case 5	($n = 3, K = 110$)		−	−	−		+	−	−		−	−	−		−	−	−		+	o	o	+	−	−	−
Case 5	($n = 5, K = 80$)	−	−	−	−	−	−	−	+	+	−	−	−	o	−	−	−	−	+	−	−	−	−	−	−
Case 5	($n = 5, K = 100$)	−	−	−	−	−	−	−	+	−	−	−	−	+	−	−	−	−	+	+	+	−	−	−	−
Case 5	($n = 5, K = 110$)		−	−	−		−	+	+		−	−	−		−	−	−		+	+	−	+	−	−	−
Case 5	($n = 10, K = 80$)		−	−	−		−	−	−		−	−	−		−	−	−		+	−	−	−	−	−	−
Case 5	($n = 10, K = 100$)		−	−	−		−	−	−		−	−	−		−	−	−		+	+	−	−	−	−	−
Case 5	($n = 10, K = 110$)			−	−			−	o			−	−			−	−			−	+	o	−	−	−
Panel B. Polynomial and European price basis																									
Case 1		−	−	−	+	−	+	−	−	−	−	+	+	−	+	+	+	o	−	+	+	−	−	+	−
Case 2		−	−	+	+	−	−	−	−	−	+	+	+	+	+	+	+	+	+	+	+	−	−	+	+
Case 3		+	+	+	+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	o	−	−	−
Case 4		o	−	+	+	o	+	−	−	+	+	+	+	o	o	+	+	+	+	+	+	−	−	+	−
Case 5	($n = 2, K = 80$)	−	−	+	−	−	−	−	−	−	+	+	+	−	+	+	+	−	−	+	+	o	−	−	−
Case 5	($n = 2, K = 100$)	−	−	−	−	−	o	−	−	−	+	+	+	o	+	+	+	−	o	+	+	+	−	−	−
Case 5	($n = 2, K = 110$)		−	−	−		−	−	−		−	+	o		o	+	+		−	+	+		+	−	−

Notes: "+" , "o" , and "–" , respectively represent good (within two standard errors from the asymptotic value), neutral (two to three standard errors away from the asymptotic value), and poor (more than three standard errors away) performance. The four columns for each method correspond to the results with different numbers of simulation paths (100, 1,000, 5,000, and 10,000 simulation paths respectively from left to the right). For case 5, n is the number of underlying assets and K is the strike price. Some cells have missing values because of not enough in-the-money paths for the specification.

Table X: Performance Comparison of Cases with Different Numbers of Exercise Times

Case		OLS				Quantile				Tikhonov				MPP				MMPP				CART			
Panel A. Case 2																									
3 exercise times	P	o	+	+	+	+	−	−	−	+	+	+	+	+	+	+	+	+	+	+	+	−	−	+	o
6 exercise times	P	−	o	+	+	o	−	−	−	o	+	+	+	+	+	+	+	o	+	+	+	−	−	+	−
3 exercise times	P and E	−	o	+	+	−	−	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	−	+	+
6 exercise times	P and E	−	o	+	+	o	−	−	−	o	+	+	+	+	+	+	+	+	+	+	+	−	−	+	+
Panel B. Case 5, $n = 2, K = 80$																									
3 exercise times	P	−	+	+	+	+	+	+	o	−	+	o	+	+	o	o	o	+	+	o	o	−	−	−	−
6 exercise times	P	−	o	+	+	o	o	+	+	−	o	+	o	o	o	o	−	o	+	o	o	+	−	−	−
12 exercise times	P	−	−	o	+	−	+	−	−	−	o	+	o	o	−	−	−	−	o	o	o	−	−	−	−
3 exercise times	P and E	−	o	+	+	−	+	o	−	o	+	+	+	+	+	+	+	+	+	+	+	o	−	−	−
6 exercise times	P and E	−	−	o	o	−	o	+	+	−	+	+	+	−	+	+	+	−	+	+	+	−	−	−	−
12 exercise times	P and E	−	−	o	−	−	−	−	−	−	+	+	+	−	+	+	+	−	o	+	+	o	−	−	−
Panel C. Case 5, $n = 2, K = 100$																									
3 exercise times	P	o	o	+	+	+	−	−	−	o	+	+	+	+	o	o	o	+	+	+	+	−	o	−	−
6 exercise times	P	−	−	+	+	+	+	−	−	−	o	+	+	+	o	−	−	+	+	+	+	−	−	−	−
12 exercise times	P	−	o	+	+	+	+	−	−	−	o	+	+	+	−	−	−	+	+	+	+	o	−	−	−
3 exercise times	P and E	−	−	o	+	−	o	−	−	o	+	+	+	o	+	+	+	o	+	+	+	o	−	−	−
6 exercise times	P and E	−	−	o	o	−	+	−	−	−	+	+	+	+	+	+	+	−	+	+	+	−	−	−	−
12 exercise times	P and E	−	−	−	o	−	o	−	−	−	+	+	+	o	+	+	+	−	+	+	+	+	−	−	−
Panel D. Case 5, $n = 2, K = 110$																									
3 exercise times	P	−	+	+	+	o	−	−	−	o	o	+	+	+	+	o	o	+	+	+	+	+	o	−	−
6 exercise times	P	o	o	+	+	+	o	−	−	o	o	+	+	+	+	−	−	+	+	+	+	+	+	−	−
12 exercise times	P	−	−	o	+	+	−	−	−	o	−	o	o	+	o	−	−	+	+	+	+	+	+	−	−
3 exercise times	P and E	−	−	+	+	−	o	−	−	o	+	+	+	+	+	+	+	o	+	+	+	+	+	−	−
6 exercise times	P and E		−	−	+		+	−	−		+	+	+		+	+	+		+	+	+		−	−	−
12 exercise times	P and E		−	−	−		+	−	−		o	+	+		+	+	+		o	+	+		+	−	−

Notes: "+" , "o", and "–", respectively represent good (within two standard errors from the asymptotic value), neutral (two to three standard errors away from the asymptotic value), and poor (more than three standard errors away) performance. "P" represents polynomial basis. "P and E" represents the basis with both polynomials and European option prices. The four columns for each method correspond to the results with different numbers of simulation paths (100, 1,000, 5,000, and 10,000 simulation paths respectively from left to the right). For case 5, n is the number of underlying assets and K is the strike price. Some cells have missing values because of not enough in-the-money paths for the specification.