# Robust Hypothesis Testing Using Wasserstein Uncertainty Sets

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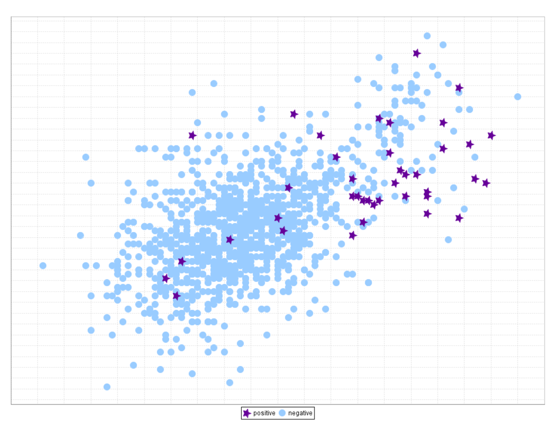
## Overview

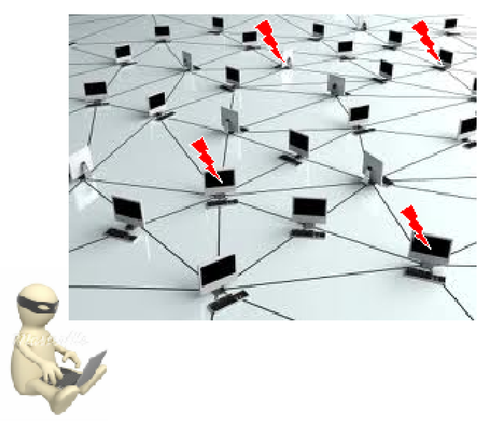
- Propose a data-driven, computationally efficient framework for non-parametric robust hypothesis testing
- Develop a nearly-optimal detector and provide an elegant interpretation of the least favorable distributions

## Introduction

- Hypothesis testing with limited samples



Imbalanced classification



Anomaly detection

- Existing robust approaches require density estimation
  - Huber's $\epsilon$-contamination sets [1]

$$\mathcal{P}_i = \{(1-\epsilon)q_i + \epsilon f_i : f_i \text{ is any probability density}\}$$

  – May not work well in high dimensions
  - Parametric hypothesis test via convex optimization by Juditsky and Nemirovski [2]

$$\mathcal{P}_i = \{P_\theta : \theta \in \Theta_i\}$$

  – Restrictive convexity assumptions on the parameters
- Goal: find a data-driven non-parametric test that is simple, optimal, robust, and scalable!

## Problem Setup

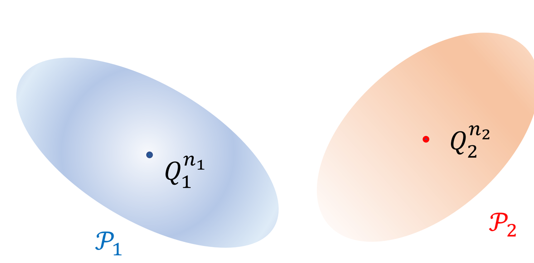
- Given an observation $\omega \in \Omega$, we would like to decide

$$H_1 : \quad \omega \sim P_1, \quad P_1 \in \mathcal{P}_1$$
$$H_2 : \quad \omega \sim P_2, \quad P_2 \in \mathcal{P}_2$$

- Wasserstein uncertainty sets:

$$\mathcal{P}_1 = \{P : \mathcal{W}(P, Q_1^{n_1}) \le \theta_1\}$$
$$\mathcal{P}_2 = \{P : \mathcal{W}(P, Q_1^{n_1}) \le \theta_2\}$$

where $Q_1^{n_1}$ and $Q_2^{n_2}$ are empirical distributions, and $\mathcal{W}$ denotes the 1-Wasserstein distance




- Problem: find a test $T : \Omega \to \{1,2\}$ to minimize the worst-case type I and type II errors

$$\inf_{T:\Omega\to\{1,2\}} \max\left\{ \sup_{P_1\in\mathcal{P}_1} P_1\{\omega : T(\omega)=2\},\ \sup_{P_2\in\mathcal{P}_2} P_2\{\omega : T(\omega)=1\} \right\}$$

- Challenges:
  - Non-convex in $T$
  - Infinite-dimensional optimization over $T$
  - Infinite-dimensional optimization over $P_1, P_2$

## Detector-based Test

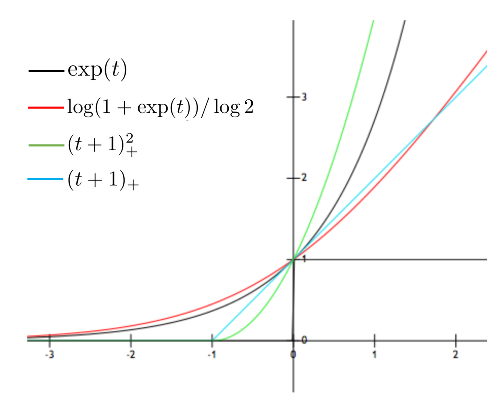
- Consider a detector $\phi$

$$\phi(\omega) \le 0 \Rightarrow T_\phi(\omega) = 1 \Rightarrow \text{Claim } H_1$$
$$\phi(\omega) > 0 \Rightarrow T_\phi(\omega) = 2 \Rightarrow \text{Claim } H_2$$

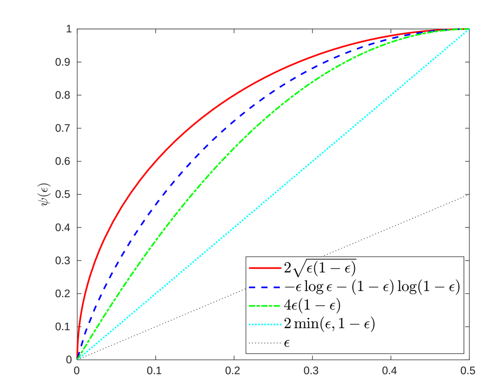
- Convex relaxation: convex surrogate $\ell$ for the 0-1 loss:

$$\inf_{\phi:\Omega\to\mathbb{R}} \sup_{P_1\in\mathcal{P}_1, P_2\in\mathcal{P}_2} \mathbb{E}_{P_1}[\ell\circ(-\phi)(\omega)] + \mathbb{E}_{P_2}[\ell\circ\phi(\omega)]$$

- *Result I* [Near-optimality]: Whenever there exists a feasible solution $T$ of the original problem with objective value $\le \epsilon \in (0, 1/2)$, there exist a feasible solution $\phi$ of the relaxed problem with objective value $\le \psi(\epsilon)$




| $\ell(t)$ | $\psi(\epsilon)$ |
|---|---|
| $\exp(t)$ | $2\sqrt{\epsilon(1-\epsilon)}$ |
| $\log(1+\exp(t))/\log 2$ | $H(\epsilon)/\log 2$ |
| $(t+1)_+^2$ | $4\epsilon(1-\epsilon)$ |
| $(t+1)_+$ | $2\epsilon$ |



## Convex Exact Reformulation

- *Result II* [Strong duality] $\inf_{\phi:\Omega\to\mathbb{R}} \sup_{P_1\in\mathcal{P}_1, P_2\in\mathcal{P}_2} \mathbb{E}_{P_1}[\ell\circ(-\phi)(\omega)] + \mathbb{E}_{P_2}[\ell\circ\phi(\omega)]$ is equivalent to

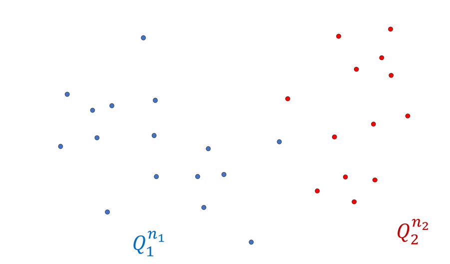
$$\max_{\substack{p_1,p_2\in\mathbb{R}_+^{n_1+n_2}\\ \gamma_1,\gamma_2\in\mathbb{R}_+^{(n_1+n_2)}\times\mathbb{R}_+^{(n_1+n_2)}}} \sum_{l=1}^{n_1+n_2} (p_1^l + p_2^l)\psi\left(\frac{p_1^l}{p_1^l+p_2^l}\right)$$

$$\text{subject to} \quad \sum_{l=1}^{n_1+n_2}\sum_{m=1}^{n_1+n_2} \gamma_k^{lm}\left\|\omega^l - \omega^m\right\| \le \theta_k,\ k=1,2,$$
$$\sum_{m=1}^{n_1+n_2} \gamma_k^{lm} = Q_k^{n_k}(\omega^l),\ 1\le l\le n_1+n_2,\ k=1,2,$$
$$\sum_{l=1}^{n_1+n_2} \gamma_k^{lm} = p_k^m,\ 1\le m\le n_1+n_2,\ k=1,2$$

- Statistical interpretation
  – Minimizing the divergence $D(P_1, P_2)$ over $P_1\in\mathcal{P}_1, P_2\in\mathcal{P}_2$ that are supported on the $n_1+n_2$ empirical points
- Complexity
  – Independent of the dimension of $\Omega$, and nearly independent of sample size $O(\ln n_1 + \ln n_2)$ for most cases
- Proof Sketch
  1. inf sup = sup inf
  – Derived using results on Wasserstein distributionally robust optimization [3]
  2. $$\inf_{\phi:\Omega\to\mathbb{R}} \mathbb{E}_{P_1}[\ell\circ(-\phi)(\omega)] + \mathbb{E}_{P_2}[\ell\circ\phi(\omega)] = \int_\Omega \psi\left(\frac{dP_1}{d(P_1+P_2)}\right)d(P_1+P_2)$$
  3. Structural characteristics of the least favorable distributions for $\sup_{P_1\in\mathcal{P}_1,P_2\in\mathcal{P}_2} \int_\Omega \psi\left(\frac{dP_1}{d(P_1+P_2)}\right)d(P_1+P_2)$

## Least Favorable Distributions and Optimal Detector

- The least favorable distributions $P_1^*, P_2^*$ have an identical support on supp $Q_1^{n_1} \cup$ supp $Q_2^{n_2}$
- The worst-case risk is $2(1 - D(P_1^*, P_2^*))$, where $P_1^*, P_2^*$ depend on $\ell$, $Q_1^{n_1}, Q_2^{n_2}$, and $\theta_1, \theta_2$

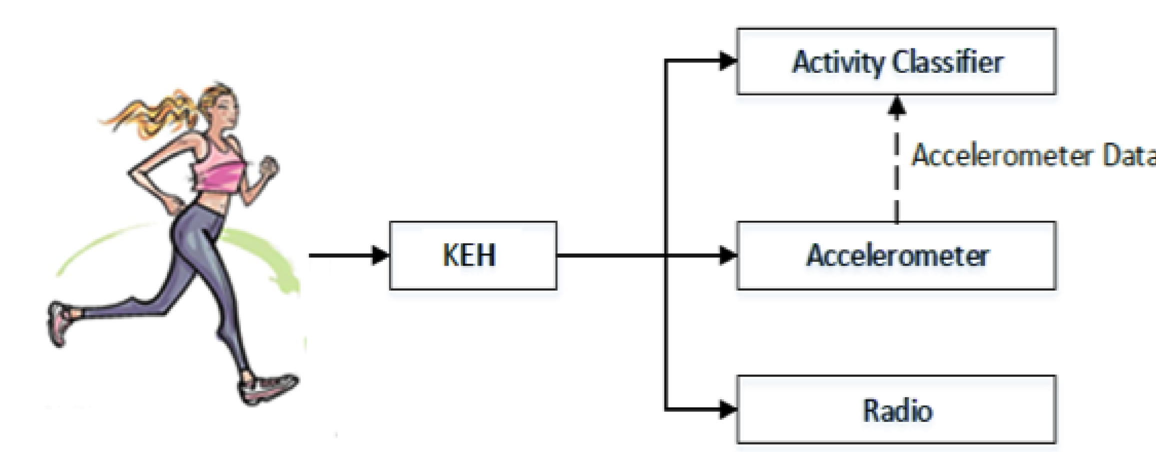



- Optimal detector

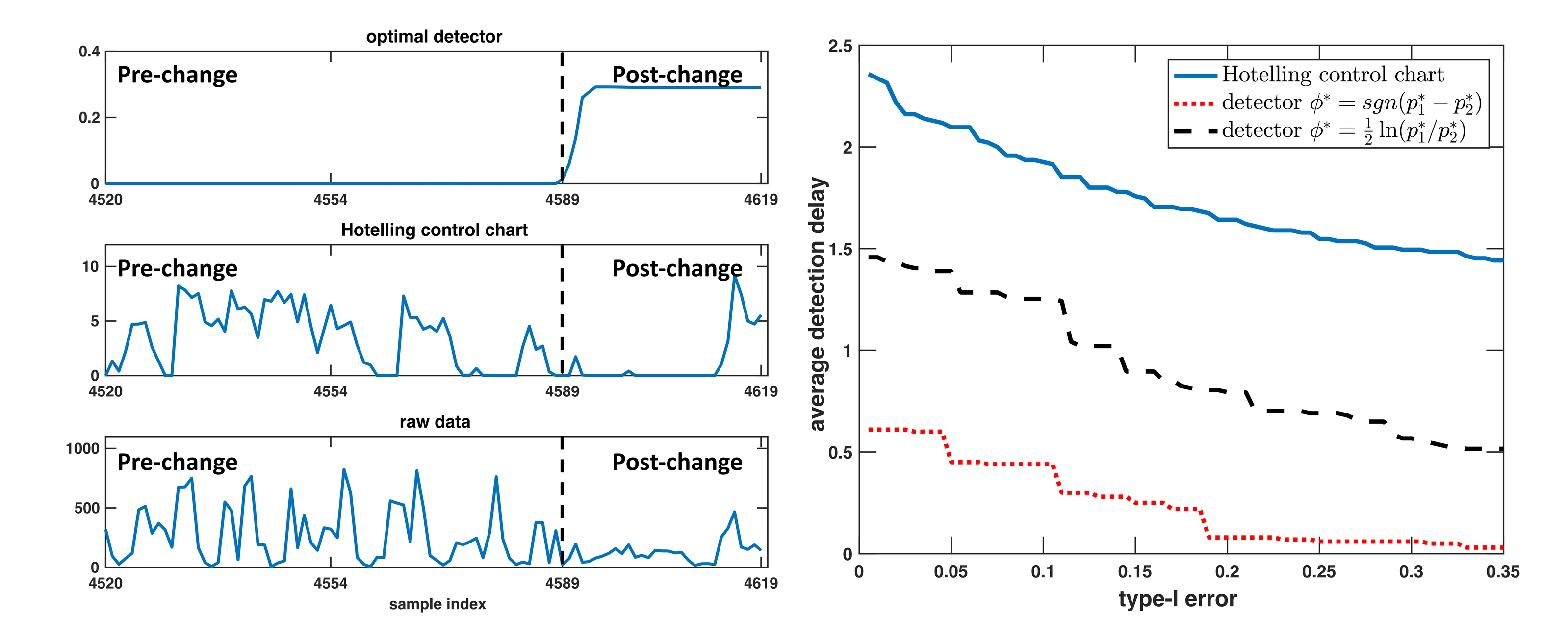
| $\ell(t)$ | $\phi(\omega)$ | $1-1/2$ Risk |
|---|---|---|
| $\exp(t)$ | $\log\sqrt{p_1^*(\omega)/p_2^*(\omega)}$ | $H^2(P_1^*, P_2^*)$ |
| $\log(1+\exp(t))/\log 2$ | $\log(p_1^*(\omega)/p_2^*(\omega))$ | $JS(P_1^*, P_2^*)/\log 2$ |
| $(t+1)_+^2$ | $1 - 2\frac{p_1^*(\omega)}{p_1^*(\omega)+p_2^*(\omega)}$ | $\chi^2(P_1^*, P_2^*)$ |
| $(t+1)_+$ | $\mathrm{sgn}(p_1^*(\omega) - p_2^*(\omega))$ | $TV(P_1^*, P_2^*)$ |

## Numerical Example

- Human activity detection




Credit: CSIRO Research




(a) (b)

Figure: Jogging vs. Walking, the average is taken over 100 sequences of data.

## References

[1] Peter J Huber. *A robust version of the probability ratio test*. The Annals of Mathematical Statistics, 36(6):1753-1758, 1965.

[2] Anatoli Juditsky Alexander Goldenshluger and Arkadi Nemirovski. *Hypothesis testing by convex optimization*. Electron. J. Statist., 9(2):1645-1712, 2015.

[3] Rui Gao and Anton Kleywegt, *Distributionally robust stochastic optimization with Wasserstein distance*. arXiv preprint arXiv:1604.02199, 2016.

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