How Often Should You Take Tactical Asset Allocation Decisions?

Byeong-Je An*
Columbia University

Andrew Ang†
Columbia University and NBER

Pierre Collin-Dufresne‡
Ecole Polytechnique Federale de Lausanne and NBER

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*Email: ba2306@columbia.edu
†Email: aa610@columbia.edu
‡Email: pierre.collin-dufresne@epfl.ch
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Abstract

About once a quarter. We compute optimal tactical asset allocation (TAA) policies over equities and bonds when both asset returns are predictable. By varying how often the weights are reset, we estimate the benefits and costs of different frequencies of TAA decisions. Tactical tilts taking advantage of predictable stock returns generate approximately twice as much value as those market-timing bond returns.
1 Introduction

Tactical asset allocation (TAA) policies aim to generate value by periodically adjusting asset class allocation targets to take advantage of time-varying expected returns. We investigate how often institutions should take such tactical decisions.\footnote{For the purposes of this article, we consider dynamic asset allocation (DAA) and TAA to be equivalent.}

We consider a long-term investor with constant relative risk aversion (CRRA) utility defined over final wealth allocating between stocks and bonds.\footnote{The simplified setting of only two assets, equities and bonds, is a reasonable first-order approximation for most institutional portfolios given that equity risk, and exposure to equity risk of many alternatives, dominates (see Leibowitz, Bova, and Hammond, 2010; Ang, 2014; among others).} Both asset classes exhibit predictability, which we calibrate to data: bond returns are time-varying and depend on the risk-free rate and yield spread, and equities can be forecasted by the same two variables as well as the dividend yield. These predictors have a long history in finance. Many researchers have used short rates and term spreads to forecast excess bond returns (see Fama and Bliss, 1987; Campbell and Shiller, 1991; among others). The dividend yield has been used to forecast equity returns at least since Dow (1920) and is intuitively appealing because high valuation ratios embed low future discount rates.

We distinguish the frequency with which an institution might change tactical targets, such as moving from a 60/40 bond/equity target to a 50/50 target, from the actual trading frequency, since the two are often different in practice. Indeed, institutional investors, such as pension and sovereign-wealth funds, often have an investment board (the principal) who chooses tactical investment targets and then delegates the actual portfolio construction to in-house or external portfolio managers (the agent).

We allow the principal to optimally switch portfolio target weights at discrete points in time. Between these times of tactical shifts, the agent continuously rebalances back to constant portfolio weights. We refer to these switching strategies as TAA because the portfolio weights are updated only infrequently. The TAA policies are optimally set for a given calendar-time updating frequency, and respond to time-varying changes in investment opportunities. As the updating intervals become more frequent, TAA weights converge to standard Merton (1971) constantly changing portfolio weights. We consider cases of optimal tactical switching for the predictable bond returns only, predictable stock returns only, and for both time-varying bond and stock returns.

We estimate the utility costs of periodically updating TAA strategies at different horizons.
compared to the optimal first-best policy of taking continuous TAA decisions (the Merton case). We show that for the quarterly and monthly frequencies, the utility losses are below 25 basis points of initial wealth as long as TAA strategies are implemented optimally. TAA programs are approximately twice as valuable for exploiting variation in the equity risk premium compared to the bond premium.

Despite a large literature on asset allocation, little is known about the impact of making TAA decisions at lower frequencies than the rebalancing frequency, because the literature typically assumes both to be the same, i.e., that principal and agent investors are identical.\(^3\) Two recent practitioner studies examining the optimal frequency of TAA decisions are Leibowitz and Bova (2011) and Almadi, Rapach, and Suri (2014).\(^4\) Neither derive the optimal portfolio strategies for predictable asset returns or for different frequencies of TAA decisions. Thus, they do not compute investors’ utility costs for suboptimal rebalancing behavior. An advantage of our framework is that the optimal time-varying TAA policies are derived for different rebalancing frequencies over predictable equity returns, bond returns, or both.

A companion appendix to this article is available online, which contains detailed derivations of the various trading strategies and computations of utility costs.

2 Model

2.1 Asset Allocation Problem

Following Merton (1969,71), Brennan, Schwartz, and Lagnado (1997) and others, we consider an investor with horizon \(T\) who maximizes constant relative risk aversion (CRRA) utility over terminal wealth:

\[
\max_{\{w(t)\}_{t=0}^{T}} \mathbb{E} \left[ \frac{W(T)^{1-\gamma}}{1-\gamma} \right],
\]

where \(\gamma\) is the investor’s degree of risk aversion, and \(w(t)\) is the weight in the investor’s portfolio held in stocks at time \(t\). We assume the remainder, \(1 - w(t)\) is held in bonds.\(^5\)

\(^3\) See Brandt (2009) and Wachter (2010) for recent summaries on asset allocation literature.

\(^4\) There are also academic studies that investigate optimal discrete rebalancing intervals in the presence of inattention costs, e.g., Abel, Eberly, and Panageas (2007) but they assume a constant opportunity set (i.e., no predictability).

\(^5\) This case is the most relevant for investors with leverage constraints, like pension funds. The case where investors can hold short or long positions in cash leads to similar results to our analysis. A disadvantage of allowing cash holdings is that the risk aversion coefficient has to be carefully calibrated, otherwise the equity premium
We write the dynamics of bonds and stocks such that the expected returns of bonds and stocks vary over time. Bond returns are predictable by short rates, \( r(t) \), and the risk-premium factor, \( y(t) \). Stock returns are also predictable by short rates and risk-premium factor, but in addition dividend yields, \( z(t) \), also have forecasting ability. Note that while the investor does not hold cash, time-varying short rates influence risk premiums of bonds and equities.

2.2 Bond Returns

We employ a two-factor version of the Vasicek (1977) term structure model with the short rate and the risk-premium factor. Stating the factor dynamics in discrete time (although the underlying model is set in continuous time, see the online appendix), we have

\[
\begin{align*}
    r(t+1) - r(t) & = \kappa_r (\bar{r} - r(t)) + \kappa_{ry} (\bar{y} - y(t)) + \sigma_r \varepsilon_r(t+1) \\
    y(t+1) - y(t) & = \kappa_y (\bar{y} - y(t)) dt + \sigma_{ry} \varepsilon_r(t) + \sigma_y \varepsilon_y(t+1),
\end{align*}
\]  

where \( \varepsilon_r(t) \) and \( \varepsilon_y(t) \) are independent and identically distributed (IID) normal random variables. The correlation between the short rate and the risk-premium factor is \( \rho_{ry} = \sigma_{ry} / \sqrt{\sigma_r^2 + \sigma_y^2} \). In equation (2), the risk premium factor influences, and is correlated with, the short rate.

We take the short rate as the three-month T-bill rate and proxy the risk-premium factor with the term spread measured as the difference between the 10-year and two-year Treasury bond yields. Since we use the short rate and term spread as state variables, bond returns reflect predictable deviations from the Expectations Hypothesis (see Dai and Singleton, 2002; Duffee, 2002). We assume a particular structure on the prices of risk allowing for unspanned dynamics in the term structure, i.e. the short rate follows one-factor version of the Vasicek (1977) under the risk-neutral measure (but is driven by two factors under the empirical measure).\(^6\)

In our empirical work, we choose total returns of 10-year Treasury constant maturity bonds to represent bond returns, \( r_B(t) \):\(^7\)

\[
r_B(t+1) = \alpha_B + \beta_{B,r} r(t) + \beta_{B,y} y(t) + \sigma_B \varepsilon_r(t+1),
\]

\(^6\) One bond is sufficient to complete bond markets in our framework, even though the unspanned risk-premium factor drives bond excess returns and thus induces non-trivial hedging demands for investors (see Duffee, 2002, and the online appendix).

\(^7\) Since the investor has a horizon of \( T \), the risk-free asset is a zero-coupon bond of \( T \) years. The 10-year maturity is a standard benchmark, tradeable, and also enables the model to be calibrated using standard Vector Autoregression (VAR) techniques as we detail below.
where the constant and coefficients on the short rate, \( r(t) \), and risk-premium factor, \( y(t) \), are determined by no-arbitrage relations and are functions of the data-generating process in equation (2) (see the online appendix).

### 2.3 Stock Returns

We build on the models of Campbell and Viceira (1999) and Stambaugh (1999) who forecast equity returns using dividend yields. In addition, we also allow short rates and term spreads to predict equity premiums, whose predictive power has been studied by Campbell (1986), Hodrick (1992), Ang and Bekaert (2007), and others. In discrete time, we assume dividend yields, \( z_t \), follow

\[
    z(t + 1) = \kappa z(t) + \sigma z \varepsilon_z(t + 1),
\]

where \( \varepsilon_z(t) \) is an IID normally distributed shock which is independent of \( \varepsilon_r(t) \) and \( \varepsilon_y(t) \). We take total returns on the S&P 500 index as stock returns, \( r_S(t) \), and dividend yields, \( z(t) \), are constructed using the sum of the previous 12 months of dividends.

Equity returns follow

\[
    r_S(t + 1) = \alpha_S + \beta_{Sr}r(t) + \beta_{Sy}y(t) + \beta_z z(t) + \sigma_S (\rho_{rs} \varepsilon_r(t + 1) + \rho_{zs} \varepsilon_z(t + 1) + \sqrt{1 - \rho_{rs}^2 - \rho_{zs}^2} \varepsilon_s(t + 1)),
\]

where \( \varepsilon_s(t) \) is an IID normally distributed shock orthogonal to \( \varepsilon_r(t) \), \( \varepsilon_y(t) \), and \( \varepsilon_z(t) \). A negative value of \( \rho_{zs} \) allows the dividend yield to be strongly negatively correlated with innovations to equity returns, as found by Stambaugh (1999). The conditional mean parameters \( (\alpha_S, \beta_{Sr}, \beta_{Sy}, \beta_z) \) can be calibrated using standard predictability regressions. We assume that stock markets are incomplete in the model by specifying time-varying prices of risk which depend on the dividend yield and the term structure factors (see the online appendix).

### 2.4 TAA Policies

We define a TAA investment policy as follows. An investor can switch his portfolio weights \( n \) times at evenly spaced points. During the period between two adjacent rebalancing dates, the investor maintains a constant portfolio weight. The weights change at rebalancing dates. We solve for the optimal TAA policy, which is a function of the number of rebalancing intervals, \( n \), the horizon of the investor, \( T \), and the state of the economy summarized by the variables that predict returns, \((r(t), y(t), z(t))\). The optimal TAA policy is time-consistent in a sense that an
investor derives the optimal policy once at time zero (using dynamic programming techniques) and the policy, which is a function of the state, remains optimal as time passes. As the intervals between rebalancing dates approach zero, or \( n \rightarrow \infty \), the solution approaches the standard continuous-time Merton (1971) case with predictable returns.\(^8\)

Figure 1 illustrates two TAA strategies for one path of simulated return predictors. For a 10-year horizon, we plot the equity weight of TAA strategies rebalanced every year \( (n = 10) \), every five years \( (n = 2) \), and the optimal Merton continuous strategy (which we label “continuous”). The TAA strategies change only at rebalancing dates, so they are step functions. As expected, the one-year TAA strategy follows the continuous strategy more closely than the TAA strategy switching every five years. It is important to note that these optimal strategies are solved at time zero and change over time as the state variables change and the horizon decreases.

3 Empirical Results

3.1 Parameter Estimates

We take monthly frequency data from January 1941 to December 2013. In our analysis, we consider systems with no predictability, predictability of bond returns only, predictability of stock returns only, and predictability in both asset classes. The continuous-time parameters are estimated by deriving the discrete-time version of the model and recovering the parameters from VAR and predictive regression coefficients (see the online appendix).

Table 1 reports the parameter estimates for a restricted VAR implied by the model (Panel A) and regressions predicting excess stock and bond returns (Panel B). Not surprisingly in Panel A, the VAR displays high persistence of the short rates, spreads, and yields. In the first line, yield spreads predict T-bill short rates echoing Campbell and Shiller (1991). This result indicates the importance of allowing for multiple factors in short rate dynamics (cf. Longstaff and Schwartz, 1992, Duffee, 2002)).

In Panel B, the point estimates in the predictive regression coefficients for excess bond returns show that when short rates and spreads are high, bond risk premiums are high. The

\(^8\) As we work in continuous time, the investor maintains constant weights in between the switching TAA decision dates by trading continuously. If the investor were to rebalance discretely, then she would buy-and-hold between two rebalancing dates. Pure discrete switching strategies do not have closed-form solutions. For simple systems with one state variable, the discrete switching strategies can be solved numerically and are close to our analytical TAA strategies. See the online appendix for further details.
coefficients, however, are not statistically significant. After the short rate process is determined, the relatively large standard errors partly reflect the difficulty in estimating time-varying prices of risk in term structure models (see Dai and Singleton, 2002). In the predictive regression for stock returns, all three variables—short rates, term spreads, and dividend yields—are jointly statistically significant. For our analysis with no predictability, and predictability only in one of stock or bond returns, we re-estimate the predictive regressions imposing these constraints. The model parameters correspondingly change for these various cases.

Figure 2 graphs the instantaneous total returns of bonds and equities implied by the model. We mark NBER recessions in the shaded areas. The implied average returns are 5.0% and 10.6% for bonds and equities which are very close to the empirical values of 5.3% and 10.7% over the sample. Expected bond returns are relatively volatile during the 1970s and early 1980s, coinciding with high inflation and monetary policy efforts to counter the high inflation during this time. Bond risk premiums are high after the 1990 and 2000 recessions, and are also very high during and after the 2008 financial crisis. Since the 1980s, expected stock returns decrease from 26.2% during 1981 to 2.1% in 2003. Recently from 2004 to 2008, equity returns average 4.9% and then increase to 2009 to 7.2% during the financial crisis. In our model, high expected returns are associated with low stock prices, which is reflected in the large, positive coefficient on the dividend yield in the equity predictive regression (equation (5) and Table 1).

3.2 Characterizing TAA Policies

Figure 3 examines how quickly the optimal TAA portfolio weights converge to continuous-time weights. We plot bond and equity holdings as a function of the rebalancing frequency. The weights are computed at time zero for an investor with a horizon of $T = 10$ years with the predictive variables set at their long-term means. We choose the risk aversion coefficient so that at time zero the optimal weights are 60% equities and 40% bonds for a TAA strategy which selects the portfolio weights once only at time zero. This level of risk aversion corresponds to Merton (1971) continuously rebalanced weights of approximately 50%, which are plotted in the dashed horizontal lines. As the switching intervals become more frequent, the TAA weights converge to the continuously rebalanced weights. In particular, TAA rebalancing at a frequency of one year or less produces TAA weights similar to the continuous weights.

In Panel B of Figure 3, the TAA equity weights are larger than the continuous strategy weights. As we will see in Figure 4, the hedging demands are negative, i.e. the continuous strategy holds less equity weight than the myopic strategy, which refers to an instantaneous
mean-variance strategy. Intuitively, the TAA strategy’s hedging are less aggressive than the continuous strategy (with the TAA producing larger hedging demands in absolute value) since under TAA, the investor does not have the ability to react to future changes in the investment opportunity set, except at discrete TAA rebalancing dates.

To examine how the optimal TAA weights change as a function of the predictive variables, Figure 4 plots the TAA equity portfolios weights together with the Merton strategy and the myopic strategy. The weights are optimal at time zero for an investor with a 10-year horizon. We consider annual TAA decisions. We vary the short rate, term spread, and dividend yield in each panel. In all panels, we change only the state variable on the $x$-axis and hold constant all other parameters and state variables.

The first notable fact in Figure 4 is that the annual TAA and continuous weights are similar, but both are significantly different from the myopic weights. This means that mean-variance strategies, which do not take into account the hedging demands induced by predictable asset returns, can result in significantly sub-optimal holdings.

Second, Figure 4 shows that equity weights are decreasing in the short rate and term spread. As can be inferred from the coefficients of the predictive regressions in Table 1, the returns of equities in excess of bonds decrease as both short rates and term spreads increase. Thus, as these predictive variables decrease, the investor favors equities over bonds. Similarly, the equity weights increase as dividend yields increase. Higher dividend yields imply higher equity risk premiums (the dividend yield does not influence the dynamics of short rates and spreads in the model in equation (2)), and the investor takes advantage of higher equity premiums by larger equity weights.

Third, the myopic weights are larger than the TAA and continuous weights. The negative hedging demands result from the investor optimally lowering the equity weight to protect her against possible negative shocks to the equity risk premium. While Campbell and Viceira (1999) found large, positive hedging demands for equity, our investor holds a 100% risky asset portfolio. Campbell and Viceira’s investor shorts the risk-free asset to fund her equity hedging demands. In our system, this role is taken by bonds: long-term equity positions are smaller than myopic weights because bond returns hedge negative shocks to expected returns better than equities. Indeed, we confirm that if we turn off the bond predictability entirely, and retain only stock predictability, then the optimal TAA position in stocks is larger than the myopic position (while the bond position is lower than its myopic counterpart).

Finally, while the TAA portfolio weights are close to the continuous Merton positions, they
react to changes in the state variables more conservatively. This can be seen in the flatter slopes of the TAA line compared to the continuously rebalanced Merton line in Panels B and C. (The effect is also there, but harder to discern, in Panel A.) This is due to TAA investors having only limited opportunities to switch their allocations. Since they cannot react instantaneously, TAA investors’ portfolios exhibit less sensitivity to changes in predictors because they recognize these opportunities are mean-reverting.

3.3 TAA Decisions at Different Frequencies

Figure 5 and Table 2 report the utility costs of TAA as a function of switching frequencies. The utility costs are stated in terms of the percentage increase of initial wealth required for an investor to be indifferent between a TAA strategy switching at a given frequency and the continuous Merton strategy. Put another way, how much does an investor need to be compensated for making TAA decisions more infrequently? We use a horizon of 10 years and set the risk aversion so that a TAA strategy choosing the portfolio weights only once at the beginning of the ten-year period yields a 60% equity and 40% bond portfolio (as in Figures 3 and 4).

Figure 5 considers the case of joint stock and bond predictability. The baseline case is the optimal Merton policy, or the TAA policy with continuous decisions. As the switching TAA intervals become more infrequent, the utility costs increase. Investors require an additional 3.0% of wealth when switching allocations twice over a 10-year horizon to have the same utility as employing the continuous Merton strategy. The utility costs are small, at 0.25% and 0.08%, for the one-quarter and one-month TAA strategies.

In Table 2, we record the case of no predictability or IID returns, and predictability in only one of stock or bond returns. The last column of Table 2 lists the joint stock and bond predictability case and is the same as Figure 5. For the IID case, the utility costs are very small across all TAA intervals; there is little benefit of taking TAA decisions more frequently because there is little value in market timing.9

Table 2 shows that the utility costs are approximately twice as large for equity predictability compared to bond predictability For example, changing portfolio weights every quarter results in utility costs of 0.14% of initial wealth in the predictable bond case compared to 0.24% for the predictable stock case. When equities are predictable, there is significantly more benefit

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9 Even when expected returns of bond and stock are constant, unexpected shocks to bond and stock returns are still correlated with state variables so hedging demands matter. The IID TAA weights are optimally computed to take into account hedging demands.
in taking TAA decisions at shorter intervals because the investor can capitalize on times when equity returns are potentially much higher than bonds. The smaller predictable variation in bond returns makes it more difficult for an investor to capitalize on bond predictability—at any TAA frequency. The value-added benefit of TAA equity decisions is particularly large for very long TAA decision intervals.

4 Conclusion

We solve for optimal TAA policies which switch target portfolio weights at periodic calendar intervals. The TAA policies are optimally computed for a long-horizon CRRA investor with time-varying expected returns. Under predictability, the optimal TAA weights are very different from the myopic, or instantaneous mean-variance, weights. We find that the utility benefits of moving to shorter decision intervals than one quarter are small and less than 25 basis points of initial wealth compared to the Merton (1971) case where TAA decisions are taken continuously.

There are at least three useful extensions to our approach. First, we do not consider variation in conditional volatilities. Given the relatively high mean reversion of volatilities, it is conceivable that TAA utility costs would still be small at intervals shorter than one quarter. Second, we ignore transaction costs in assuming that the agent can rebalance continuously in between TAA decision dates. While the literature started by Constantinides (1983) shows that closed-form solutions for portfolio weights in a non-IID environment are rarely available, we expect that the presence of transaction costs would only serve to lengthen the optimal TAA decision interval. On the other hand, the small transaction costs for implementing overlay TAA strategies with future contracts would likely not change our results. Lastly, we have focused only on the TAA decision between two asset classes: equity and bonds. It would be natural to extend this analysis to more asset classes, such as commodities, inflation protected bonds, and real estate.
References


### Table 1: Vector Autoregression and Predictive Regressions

<table>
<thead>
<tr>
<th></th>
<th>Constant</th>
<th>Short Rate</th>
<th>Yield Spread</th>
<th>Dividend Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: VAR for the Short Rate, Yield Spread, and Dividend Yield</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Short Rate</td>
<td>0.0006</td>
<td>0.9845</td>
<td>-0.0239</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(0.0001)**</td>
<td>(0.0240)**</td>
<td>(0.0017)**</td>
<td></td>
</tr>
<tr>
<td>Yield Spread</td>
<td>0.0002</td>
<td>-</td>
<td>0.9766</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(0.0001)**</td>
<td></td>
<td>(0.0087)**</td>
<td></td>
</tr>
<tr>
<td>Dividend Yield</td>
<td>0.0007</td>
<td>-</td>
<td>-</td>
<td>0.9826</td>
</tr>
<tr>
<td></td>
<td>(0.0002)**</td>
<td></td>
<td></td>
<td>(0.0059)**</td>
</tr>
<tr>
<td><strong>Panel B: Predictability Regressions</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Excess Bond Returns</td>
<td>-0.0020</td>
<td>0.0340</td>
<td>0.2653</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(0.0007)**</td>
<td>(0.1223)</td>
<td>(0.3457)</td>
<td></td>
</tr>
<tr>
<td>Excess Stock Returns</td>
<td>-0.0028</td>
<td>0.0055</td>
<td>0.0108</td>
<td>0.2235</td>
</tr>
<tr>
<td></td>
<td>(0.0036)</td>
<td>(0.0026)*</td>
<td>(0.0007)**</td>
<td>(0.0910)*</td>
</tr>
</tbody>
</table>

Panel A reports coefficients of a discrete-time restricted Vector Autoregression (VAR) implied by the model in equations (2) and (5). Panel B reports coefficients of predictive regressions for excess stock and bond returns. All parameters are estimated at the monthly frequency with data from January 1941 to December 2013. Standard errors are in parentheses. We denote 5% and 1% levels of significance by * and **, respectively.
The table reports utility costs of TAA strategies switching at set periodic calendar intervals versus the continuous Merton strategy which switches instantaneously. Utility costs are reported in percentage terms of initial wealth and represent the increase in initial wealth required to make the investor with a given TAA strategy have the same utility if she had the ability to implement the optimal Merton strategy with predictable returns. We take an investor with a 10-year horizon and risk aversion of 7.9, which corresponds to a 60% equity and 40% bond portfolio at time zero for the TAA strategy which fixes the portfolio weights only once at time zero (the 10-year switching TAA strategy). We compute the utility costs integrating across the steady-state distribution of the state variables. The various cases correspond to non-predictable asset returns, only bond returns are predictable, only stock returns are predictable, or both bond and stock returns are predictable.

<table>
<thead>
<tr>
<th>Switching Interval</th>
<th>IID Returns</th>
<th>Predictable Bond Only</th>
<th>Predictable Stock Only</th>
<th>Predictable Stocks and Bonds</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 Years</td>
<td>0.42</td>
<td>2.38</td>
<td>4.76</td>
<td>4.70</td>
</tr>
<tr>
<td>5 Years</td>
<td>0.17</td>
<td>1.35</td>
<td>3.16</td>
<td>3.03</td>
</tr>
<tr>
<td>2 Years</td>
<td>0.05</td>
<td>0.81</td>
<td>1.61</td>
<td>1.62</td>
</tr>
<tr>
<td>1 Year</td>
<td>0.02</td>
<td>0.48</td>
<td>0.90</td>
<td>0.92</td>
</tr>
<tr>
<td>6 Months</td>
<td>0.01</td>
<td>0.26</td>
<td>0.48</td>
<td>0.49</td>
</tr>
<tr>
<td>1 Quarter</td>
<td>0.01</td>
<td>0.14</td>
<td>0.24</td>
<td>0.25</td>
</tr>
<tr>
<td>1 Month</td>
<td>0.00</td>
<td>0.05</td>
<td>0.08</td>
<td>0.08</td>
</tr>
</tbody>
</table>
The figure plots the portfolio weights in equity of three different trading strategies: the five-year TAA, the one-year TAA, and the optimal continuous-time Merton (1971) weights (“continuous”) for an investor with a $T = 10$ year horizon over one simulated path of state variables.
The figure plots the instantaneous expected log returns of bonds and equities implied by the model. NBER recessions are shaded.
The figure plots portfolio weights of TAA and the Merton (1971) continuously rebalanced strategy at time 0 for an investor with a horizon of $T = 10$ years, risk aversion of 7.9, with steady-state values of the predictive variables. The optimal TAA weights in the solid line are a function of the rebalancing frequency. The optimal Merton (1971) continuous-time weights are drawn in the horizontal dashed line.
Figure 4: TAA Equity Portfolio Weights vs. Predictive Variables

Panel A: Short Rates

Panel B: Term Spreads

Panel C: Dividend Yields
Note to Figure 4
We plot equity weights at time zero for the optimal TAA strategy switching allocations every year, the strategy with continuous rebalancing, and the myopic strategy. The last is valid for an instantaneous mean-variance portfolio. The three rows correspond to short rate, term spreads, and dividend yield predictors. Vertical lines indicate the steady-state mean of each state variable. In all panels, we vary only the state variable on the $x$-axis and hold constant all other parameters and state variables. We use an horizon of $T = 10$ years and a risk aversion of 7.9.
We plot the utility costs of taking TAA decisions at various intervals relative to the continuous Merton (1971) strategy which switches instantaneously. The utility costs are integrated over the steady distribution of state variables. The utility costs are computed for a 10-year horizon and a risk aversion of 7.9, which corresponds to a 60% equity and 40% bond portfolio at time zero for the TAA strategy which chooses the portfolio weights only once at the beginning of the ten-year period (the 10-year TAA strategy). We consider the full case of predictability in both stocks and bonds.
Internet Appendix for
How Often Should You Take
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Byeong-Je An*
Columbia University

Andrew Ang†
Columbia University and NBER

Pierre Collin-Dufresne‡
Ecole Polytechnique Federale de Lausanne and NBER

This Version: March 5, 2015

*Email: ba2306@columbia.edu
†Email: aa610@columbia.edu
‡Email: pierre.collin-dufresne@epfl.ch
A Bond Returns

We employ a two-factor version of the Vasicek (1977) term structure model with short rates, \( r(t) \), and a risk-premium factor, \( y(t) \), as state variables:

\[
\begin{align*}
    dr(t) &= \left[ \kappa_r \{ \bar{r} - r(t) \} + \kappa_{ry} \{ \bar{y} - y(t) \} \right] dt + \sigma_r dZ_r(t) \\
    dy(t) &= \kappa_y (\bar{y} - y(t)) dt + \sigma_{ry} dZ_r(t) + \sigma_y dZ_y(t),
\end{align*}
\]

where \( Z_r \) and \( Z_y \) are independent Brownian motions. The correlation between the short rate and term spread factor is \( \rho_{ry} = \sigma_{ry}/\sqrt{\sigma^2_{ry} + \sigma^2_y} \). We take the short rate as the three-month T-bill rate and proxy the risk-premium factor with the term spread measured as the difference between the 10-year and two-year Treasury bond yields.

We specify a price of risk-free rate risk as an affine function of state variables:

\[
\Lambda_r(t) = \lambda_r + \phi_r r(t) + \phi_y y(t). 
\]

Further, we assume that

\[
\kappa_{ry} = -\sigma_r \phi_y. 
\]

This restriction effectively makes the risk-free rate follow a one-factor Vasicek term structure model under the risk-neutral measure. In other words, the yield spread shock is not spanned in bond markets (see Collin-Dufresne and Goldstein, 2002).

The dynamics of the risk-free rate under the risk-neutral measure is

\[
\begin{align*}
    dr(t) &= \left[ \kappa_r \{ \bar{r} - r(t) \} + \kappa_{ry} \{ \bar{y} - y(t) \} \right] dt + \sigma_r (dZ^Q_r(t) - \Lambda_r(t)) \\
    &= \kappa^Q_r (\bar{r}^Q - r(t)) dt + \sigma_r dZ^Q_r(t),
\end{align*}
\]

where \( \bar{r}^Q = \frac{\kappa_r \bar{r} + \kappa_{ry} \bar{y} - \sigma_r \Lambda_r}{\kappa^Q_r} \), and \( \kappa^Q_r = \kappa_r + \sigma_r \phi_r \). Also, \( Z^Q_r \) represents a Brownian motion under the risk-neutral measure. Note that the mean-reverting speed and mean level differ in two measures. Now, the time \( t \) price of zero coupon bond maturing at time \( T \geq t \) can be derived:

\[
\begin{align*}
    P(t, T) &= \exp \left( A_1(T-t) + A_2(T-t)r(t) \right) \\
    A_2(\tau) &= -1 - e^{-\kappa^Q_r \tau} \\
    A_1(\tau) &= -\left( r^Q - \frac{\sigma^2_r}{2\kappa^Q_r} \right) (A_2(\tau) + \tau) - \frac{\sigma^2_r A_2(\tau)^2}{4\kappa^Q_r}. 
\end{align*}
\]

In the utility maximization problem we consider later, an investor who takes bond and stock prices as given and allocates his wealth only in constant time-to-maturity bond index and stock,
not cash. Thus, we need to specify the process of the constant time-to-maturity bond index. We
assume without loss of generality that the time-to-maturity of bond index the investor can trade
is $T$, which is same as the investment horizon.

Denote $B(t)$ as the index value of $T$ time-to-maturity bond index. Then, the following holds

$$
\frac{dB(t)}{B(t)} = \frac{dP(t, s)}{P(t, s)} \bigg|_{s=t+T} + \frac{1}{P(t, s)} \frac{\partial P(t, s)}{\partial s} \bigg|_{s=t+T} \equiv \mu_B(r(t), y(t)) dt + A_2(T) \sigma_r dZ_r(t),
$$

where

$$
\mu_B(r(t), y(t)) = \alpha_B + \beta_{B,r} r(t) + \beta_{B,y} y(t) \quad (10)
$$

$$
\alpha_B = a_1(T) + A_2(T) \sigma_r \lambda_r \quad (11)
$$

$$
\beta_{B,r} = 1 + a_2(T) + A_2(T) \sigma_r \phi_r \quad (12)
$$

$$
\beta_{B,y} = A_2(T) \sigma_r \phi_y. \quad (13)
$$

and $a_i(T) = \frac{\partial A_i(s-t)}{\partial s} \bigg|_{s=t+T}$ for $i = 1, 2$. The return of the $T$ time-to-maturity bond index
consists of two parts: the return from holding $P(t, t + T)$ between time $t$ and $t + dt$ and the
rollover return from selling $P(t + dt, t + T)$ and buying $P(t + dt, t + dt + T)$.

## B Stock Returns

We build on the models of Campbell and Viceira (1999) and Stambaugh (1999) who forecast
equity returns using dividend yields. In addition, we also allow short rates and term spreads
to predict equity premiums, whose predictive power has been studied by Campbell (1986),
Hodrick (1992), Ang and Bekaert (2007), and others. We assume dividend yields follow

$$
dz(t) = \kappa_z \left( z - z(t) \right) + \sigma_z dZ_z(t),
$$

where $Z_z$ is a Brownian motion, which is independent of $Z_r$ and $Z_y$. We also specify a price of
equity specific risk $\Lambda_s(t)$ as an affine function of dividend yields:

$$
\Lambda_s(t) = \lambda_s + \phi_s z(t). \quad (15)
$$

Under the risk-neutral measure, equity returns can be written as

$$
\frac{dS(t)}{S(t)} = r(t) dt + \sigma_s \left( \rho_{rs} dZ_r^Q(t) + \rho_{zs} dZ_z^Q(t) + \sqrt{1 - \rho_{rs}^2 - \rho_{zs}^2} dZ_s^Q(t) \right), \quad (16)
$$
where $Z_Q^s$ is a Brownian motion under the risk-neutral measure, which is independent of $Z_r^Q$, $Z_y^Q$, and $Z_z^Q$. Note that $Z_z^Q = Z_z$, i.e. the stock market is incomplete, and the expected return under the risk-neutral measure is just the short rate.

Under the physical measure, equity returns follow

$$\frac{dS(t)}{S(t)} = r(t)dt + \sigma_s \rho_{rs} (dZ_r(t) + \Lambda_r(t))$$

$$+ \sigma_s \rho_{zs} dZ_z(t) + \sigma_s \sqrt{1 - \rho^2_{rs} - \rho^2_{zs}} (dZ_s(t) + \Lambda_s(t))$$

$$= \mu_S (r(t), y(t), z(t)) dt + \sigma_s \left( \rho_{rs} dZ_r(t) + \rho_{zs} dZ_z(t) + \sqrt{1 - \rho^2_{rs} - \rho^2_{zs}} dZ_s(t) \right),$$

where

$$\mu_S (r(t), y(t), z(t)) = \alpha_S + \beta_{Sr} r(t) + \beta_{Sy} y(t) + \beta_z z(t)$$

$$\alpha_S = \sigma_s \left( \rho_{rs} \lambda_r + \sqrt{1 - \rho^2_{rs} - \rho^2_{zs}} \lambda_s \right)$$

$$\beta_{Sr} = 1 + \sigma_s \rho_{rs} \phi_r$$

$$\beta_{Sy} = \sigma_s \rho_{rs} \phi_y$$

$$\beta_z = \sigma_s \sqrt{1 - \rho^2_{rs} - \rho^2_{zs}} \phi_s.$$ 

### C Asset Allocation

Following Brennan, Schwartz, and Lagnado (1997) and others, we consider an investor with horizon $T$ who maximizes CRRA utility over terminal wealth:

$$\max_{\{w(t)\}_{t=0}^T} \mathbb{E} \left[ \frac{W(T)^{1-\gamma}}{1-\gamma} \right],$$

where $\gamma$ is the investor’s degree of risk aversion, and $w(t)$ is the weight in the investor’s portfolio held in stocks at time $t$. We assume the remainder, $1 - w(t)$, is held in bonds.

The wealth process follows

$$\frac{dW(t)}{W(t)} = (\mu_B(t) + w(t) (\mu_S(t) - \mu_B(t))) dt + (\sigma_B + w(t) \sigma_{S*} ) dZ(t)$$

where $dZ(t) = [dZ_r(t) \ dZ_y(t) \ dZ_z(t) \ dZ_s(t)]$, and

$$\sigma_B = \begin{bmatrix} A_2(T) \sigma_r & 0 & 0 \\ \sigma_s \rho_{rs} - A_2(T) \sigma_r & \sigma_s \sqrt{1 - \rho^2_{rs} - \rho^2_{zs}} \end{bmatrix}$$

$$\sigma_{S*} = \begin{bmatrix} \sigma_s \rho_{rs} - A_2(T) \sigma_r & \sigma_s \sqrt{1 - \rho^2_{rs} - \rho^2_{zs}} \\ \sigma_s \rho_{rs} & \sigma_s \sqrt{1 - \rho^2_{rs} - \rho^2_{zs}} \end{bmatrix}.$$
C.1 Continuous Merton (1971) Policy

We take the continuous Merton (1971) policy as a benchmark. This policy continuously rebalances to fixed portfolio weights when returns are IID, and continuously switches to time-varying portfolio weights when returns are predictable.

We use the stochastic control approach to solve the problem. Let \( J(W, X, t) \) denote the indirect utility function. The principle of optimality leads to the following Hamilton-Jacobi-Bellman equation for \( J \):

\[
\max_{w(t)} J_t + \mathcal{L} J = 0, \tag{28}
\]

where

\[
\mathcal{L} J = J_W (\mu_B + w(\mu_B - \mu_S)) + J_X K (\theta - X) + J_{WX} W \sigma_X (\sigma_B + w\sigma_S^*)^\top + \frac{1}{2} J_{WW} W^2 (\sigma_B + w\sigma_S^*) (\sigma_B + w\sigma_S^*)^\top + \frac{1}{2} \text{tr} (J_{XX} \sigma_X \sigma_X^\top), \tag{29}
\]

with boundary condition

\[
J(W(T), X(T), T) = \frac{W(T)^{1-\gamma}}{1-\gamma}. \tag{30}
\]

The coefficients \( K, \theta, \) and \( \sigma_X \) can be obtained by stacking all three state variables, \( X = [r(t) y(t) z(t)]^\top \). The indirect utility function \( J \) is conjectured to have the form:

\[
J(W(t), X(t), t) = \frac{W(t)^{1-\gamma}}{1-\gamma} F(X(t), t)^\gamma. \tag{31}
\]

Under this conjecture, the optimal portfolio weight of stock is given by

\[
w^*(t) = \frac{\mu_S - \mu_B}{\gamma\sigma_S^*\sigma_S^\top} - \frac{\sigma_B \sigma_S^*}{\sigma_S^*\sigma_S^*} + \frac{F_X \sigma_X \sigma_X^\top}{F \sigma_S^*\sigma_S^*}. \tag{32}
\]

We can interpret the optimal portfolio weight as two parts: the myopic demand and the hedging demand. The first two terms in equation (32) represent the myopic demand. The term \( (\mu_S - \mu_B)/(\gamma\sigma_S^*\sigma_S^\top) \) is the standard formula for an IID environment with a constant risk premium. In our setting, the risk-free rate changes over time, so the investor also cares about the covariance of stock and bond returns represented in the second term, \( (\sigma_B \sigma_S^\top)/(\sigma_S^*\sigma_S^\top) \). The last term in equation (32) is the hedging demand, which allows the investor to hedge possible future variation of the state variables by holding an off-setting position in assets whose return is correlated with those state variables.

To solve for \( F(X(t), t) \), we conjecture its form and then verify. Our conjecture is that

\[
F(X(t), t) = \exp \left( B_1(\tau) + B_2(\tau) X(t) + \frac{1}{2} X(t)^\top B_3(\tau) X(t) \right), \tag{33}
\]

\]
where \( \tau = T - t \) and the matrix \( B_3 \), the vector \( B_2 \), and the scalar \( B_1 \) satisfy a system of ordinary differential equations (ODEs). Substituting the optimal portfolio weight into equation (28) gives us the following partial differential equation (PDE):

\[
F_t + F \left( \frac{1 - \gamma}{2\gamma^2} (\Lambda^* - \gamma \sigma_B^*)^\top \sigma_S^\top, (\sigma_S^* \sigma_S^\top)^{-1} \sigma_S^* (\Lambda^* - \gamma \sigma_B^*) + \frac{1 - \gamma}{\gamma} r^* \right)
+ F_X \left[ K (\theta - X) + \frac{1 - \gamma}{\gamma} \sigma_X \sigma_S^* (\sigma_S^* \sigma_S^\top)^{-1} \sigma_S^* (\Lambda^* - \gamma \sigma_B^*) + (1 - \gamma) \sigma_X \sigma_B^* \right]
+ \frac{\gamma - 1}{2F} F_X \left[ \sigma_X \sigma_X^\top - \sigma_X \sigma_S^* (\sigma_S^* \sigma_S^\top)^{-1} \sigma_S^* \sigma_X^\top \right]
F_X + \frac{1}{2} \text{tr} \left( F_X \sigma_X \sigma_X^\top \right) = 0,
\]

where \( \Lambda^* = \lambda^* + \phi^* X \) such that

\[
\sigma_S^* \Lambda^* = \mu_S - \mu_B,
\]

and \( r^* = \delta_0^* + \delta_1^* X \) such that

\[
r^* = \mu_B - \frac{\gamma}{2} \sigma_B^* \sigma_B^\top.
\]

Plugging equation (33) into the PDE and matching coefficients on \( X(t)^\top [] X(t), X(t) \), and the constant term leads us to a system of ODEs:

\[
\dot{B}_3(\tau) = 2B_3 P + B_3 Q B_3^\top + \frac{1 - \gamma}{\gamma^2} \phi^* \sigma_S^\top \sigma_S^\top (\sigma_S^* \sigma_S^\top)^{-1} \sigma_S^* \phi^*
\]

\[
\dot{B}_2(\tau) = B_2 P + B_2 Q B_3^\top + RB_3^\top
+ \frac{1 - \gamma}{\gamma^2} (\lambda^* - \gamma \sigma_B^*)^\top \sigma_S^\top (\sigma_S^* \sigma_S^\top)^{-1} \sigma_S^* \phi^* + \frac{1 - \gamma}{\gamma} \delta_1^*
\]

\[
\dot{B}_1(\tau) = B_2 R + \frac{1}{2} B_2 Q B_2^\top + \frac{1}{2} \text{tr} \left( B_3 \sigma_X \sigma_X^\top \right)
+ \frac{1 - \gamma}{2\gamma^2} (\lambda^* - \gamma \sigma_B^*)^\top \sigma_S^\top (\sigma_S^* \sigma_S^\top)^{-1} \sigma_S^* (\lambda^* - \gamma \sigma_B^*) + \frac{1 - \gamma}{\gamma} \delta_0^*,
\]

where

\[
P = -K + \frac{1 - \gamma}{\gamma} \sigma_X \sigma_S^\top (\sigma_S^* \sigma_S^\top)^{-1} \sigma_S^* \phi^*
\]

\[
Q = \gamma \sigma_X \sigma_X^\top + (1 - \gamma) \sigma_X \sigma_S^* (\sigma_S^* \sigma_S^\top)^{-1} \sigma_S^* \sigma_X^\top
\]

\[
R = K \theta + \frac{1 - \gamma}{\gamma} \sigma_X \sigma_S^* (\sigma_S^* \sigma_S^\top)^{-1} \sigma_S^* (\lambda^* - \gamma \sigma_B^*) + (1 - \gamma) \sigma_X \sigma_B^*.
\]

The boundary condition is

\[
B_1(T) = B_2(T) = B_3(T) = 0.
\]

### C.2 Myopic Policy

We introduce a myopic policy which ignores the hedging demands present in the continuous Merton policy. A myopic investor times the market over the next (instantaneous) period and has
a portfolio weight represented by the first two terms in equation (32). The portfolio weights can be expressed as

\[ w(t) = \alpha_0 + \alpha_1 X(t), \]

where

\[
\alpha_0 = \frac{\sigma_s^* (\lambda^* - \gamma \sigma_B^T)}{\gamma \sigma_s^* \sigma_{S^*}^T} \quad (45)
\]

and

\[
\alpha_1 = \frac{\sigma_s^* \phi^*}{\gamma \sigma_s^* \sigma_{S^*}^T}. \quad (46)
\]

To calculate utility costs, we solve for the indirect utility when the investor follows the above strategy. Denote \( \hat{J}(W, X, t) \) as the indirect utility corresponding to \((\alpha_0, \alpha_1)\). Then \( \hat{J}(W, X, t) \) should also satisfy equation (28). Since \( w(t) \) is linear in \( X(t) \), \( \hat{J}(W, X, t) \) takes the same form as the continuous Merton policy:

\[
\hat{J}(W(t), X(t), t) = \frac{W(t)^{1-\gamma}}{1-\gamma} \hat{F}(X(t), t)\gamma \]

Similarly, we conjecture that \( \hat{F}(X, t) \) is exponential quadratic:

\[
\hat{F}(X(t), t) = \exp \left( \hat{B}_1(\tau) + \hat{B}_2(\tau)X(t) + \frac{1}{2} X(t)^\top \hat{B}_3(\tau)X(t) \right), \quad (48)
\]

where \( \hat{B}_1, \hat{B}_2, \) and \( \hat{B}_3 \) satisfy a system of ODEs.

With a similar procedure as solving the continuous Merton policy in the previous section, we obtain the system of ODEs:

\[
\dot{\hat{B}}_3(\tau) = 2\hat{B}_3 \hat{P} + \hat{B}_3 \hat{Q} \hat{B}_3^\top + \frac{2(1-\gamma)}{\gamma} \left\{ \alpha_1^\top \sigma_s^* \phi^* - \gamma \frac{1}{2} \alpha_1^\top \sigma_s^* \sigma_{S^*}^\top \alpha_1 \right\} \quad (49)
\]

\[
\dot{\hat{B}}_2(\tau) = \hat{B}_2 \hat{P} + \hat{B}_2 \hat{Q} \hat{B}_2^\top + \hat{R} \hat{B}_2^\top + \frac{1-\gamma}{\gamma} \left\{ \alpha_0^\top \sigma_s^* \phi^* + \lambda^* \sigma_{S^*}^\top \alpha_1 - \gamma (\sigma_B + \alpha_0 \sigma_{S^*}) \sigma_{S^*}^\top \alpha_1 + \delta_0^* \right\} \quad (50)
\]

\[
\dot{\hat{B}}_1(\tau) = \hat{B}_2 \hat{R} + \frac{1}{2} \hat{B}_2 \hat{Q} \hat{B}_2^\top + \frac{1}{2} \tr \left( \hat{B}_3 \sigma_X \sigma_X^\top \right) + \frac{1-\gamma}{\gamma} \left\{ \alpha_0^\top \sigma_s^* \lambda^* - \gamma (\sigma_B + \alpha_0 \sigma_{S^*}) (\sigma_B + \alpha_0 \sigma_{S^*})^\top + \delta_0^* \right\}, \quad (51)
\]

where

\[
\hat{P} = -K + (1-\gamma) \sigma_X \sigma_{S^*}^\top \alpha_1 \quad (52)
\]

\[
\hat{Q} = \gamma \sigma_X \sigma_X^\top \quad (53)
\]

\[
\hat{R} = K\theta + (1-\gamma) \sigma_X (\sigma_B + \alpha_0 \sigma_{S^*})^\top, \quad (54)
\]

and \( \delta_0^*, \delta_1^* \) are such that \( \mu_B = \delta_0^* + \delta_1^* X \). The boundary conditions are

\[
\hat{B}_1(T) = \hat{B}_2(T) = \hat{B}_3(T) = 0. \quad (55)
\]
C.3 Tactical Asset Allocation Policy

We define a TAA investment policy as follows. An investor can switch his portfolio weights \( n \) times at evenly spaced points. During the period between two adjacent rebalancing dates, the investor maintains a constant portfolio weight. The weights change at a rebalancing date. We solve for the optimal TAA policy, which is a function of the number of rebalancing intervals, \( n \), the horizon of the investor, \( T \), and the state of the economy summarized by the variables that predict returns, \((r(t), y(t), z(t))\). As the intervals between rebalancing approach zero, or \( n \to \infty \), then we approach the standard continuous-time Merton (1971) case with predictable returns.

Let \( \hat{J}_k(W_k, X_k; n) \) be the value function at \( k \)-th switching date of TAA switching \( n \) times, and take the following form:

\[
\hat{J}_k(W_k, X_k; n) = \frac{W_k^{1-\gamma}}{1-\gamma} \hat{F}_k(X_k; n)^\gamma \tag{56}
\]

\[
\hat{F}_k(X_k; n) = \exp \left( \hat{B}_{1,k} + \hat{B}_{2,k} X_k + \frac{1}{2} X_k^\top \hat{B}_{3,k} X_k \right). \tag{57}
\]

We refer to time 0 as the zero-th switching date. Then, the recursion formulas for \( \hat{B}_1 \), \( \hat{B}_2 \), and \( \hat{B}_3 \) of the TAA policy are equations (49), (50), and (51), but \( \alpha_0 \) is undetermined and \( \alpha_1 = 0 \). Now, the question is what is the optimal \( \alpha_0 \)? The agent chooses \( \alpha_0 \) to maximize the value function at the \( k \)-th rebalancing date. The FOC with respect to \( \alpha_0 \) is

\[
\frac{\partial \hat{B}_{1,k}}{\partial \alpha_0} + \frac{\partial \hat{B}_{2,k}}{\partial \alpha_0} X_t = 0. \tag{58}
\]

Note that \( \hat{B}_{3,k} \) does not depend on \( \alpha_0 \). The above equation tells us that the optimal target portfolio weight at \( k \)-th rebalancing date is linear in the state variables:

\[
\alpha_0^* = c_{0,k} + c_{1,k} X_k. \tag{59}
\]

We substitute \( \alpha_0^* \) in \( \hat{B}_{1,k} \) and \( \hat{B}_{2,k} \), and re-collect coefficients on constant, \( X_k \), and \( X_k \cdot X_k \). Then, we have a new \( \hat{B}_{1,k} \) and \( \hat{B}_{2,k} \). We repeat this procedure until time 0 to obtain the optimal target portfolio weight at each TAA decision point and the value function at time 0.

D Utility Cost

We measure the utility costs in terms of the percentage increase of initial wealth required for an investor to be indifferent between a TAA strategy switching at a given frequency and the
continuous Merton strategy. We can compute \( F(X(0), 0) \) in equation (33) when the continuous Merton policy is employed, and \( \tilde{F}_0(X(0); n) \) in equation (57) when the TAA with \( n \)-switching is used. Then, the utility cost of TAA switching \( n \) times is calculated as

\[
c(X(0); n) = \left( \frac{F(X(0), 0)}{\tilde{F}_0(X(0); n)} \right)^{\frac{1}{1-n}} - 1. \tag{60}
\]

Note that the utility cost depends on the initial value of state variables. To find the utility cost which is independent of the initial state variables, we numerically integrate over the initial state variables using the stationary distribution.

\section*{E Estimation}

We take monthly frequency data from January 1941 to December 2013. In our analysis, we consider systems with no predictability, predictability of bond returns only, only predictable stock returns, and when expected returns of both assets vary over time. The continuous-time parameters are estimated by deriving the discrete-time version of the model and recovering the parameters from VAR and predictive regression coefficients.

Table A-1 provides summary statistics of the state variables and excess returns. All numbers are annualized. Note that the short rate and yield spread are negatively correlated. The historical excess bond return is 1.35% and the excess equity return is 6.72%. The volatilities of excess bond and stock returns are 6.83% and 14.55%, respectively. The mean and volatilities translate into Sharpe ratios of 0.20 for bonds and 0.46 for equities.

Let \( \hat{X} \) be the augmented state variables vector: \( \hat{X}(t) = [r(t) \ y(t) \ z(t) \ \log B(t) \ \log S(t)]^\top \). Then, our model can be written

\[
d\hat{X}(t) = \left( \hat{\mu} + \hat{K}\hat{X}(t) \right) dt + \hat{\sigma}dZ(t), \tag{61}
\]

where \( \hat{\mu}, \hat{K}, \) and \( \hat{\sigma} \) are vector representations of the parameters of each variable in \( \hat{X} \). The discrete-time process implied by the above continuous-time mean-reverting process is

\[
\hat{X}(t + \Delta t) = \left( \int_t^{t+\Delta t} e^{\hat{K}(t+\Delta t-s)} ds \right) \hat{\mu} + e^{\hat{K}\Delta t} \hat{X}(t) + \int_t^{t+\Delta t} e^{\hat{K}(t+\Delta t-s)} \hat{\sigma}dZ(t), \tag{62}
\]

where \( e^{\hat{K}\Delta t} \) is a matrix exponential. The variance-covariance matrix \( \Sigma \) is

\[
\Sigma = \int_t^{t+\Delta t} e^{\hat{K}(t+\Delta t-s)} \hat{\sigma} \hat{\sigma}^\top e^{\hat{K}(t+\Delta t-s)} ds. \tag{63}
\]
This is a restricted Vector Autoregression (VAR).

We estimate equation (62) and report the coefficients in Table 1 of the paper. Table 1 reports the parameter estimates for the restricted VAR implied by the model (Panel A) and regressions predicting excess stock and bond returns (Panel B). We recover the continuous-time parameters from the discrete-time VAR estimates, which we report in Table A-2. Panel D corresponds to the case that both bond and stock returns are predictable (recovered from Table 1 in the main paper). The other cases are:

- For IID returns, we set $\phi_r = \phi_y = \phi_s = 0$, i.e. the prices of risks are constant.
- For the case of predictable bond returns only, we set $\phi_s = \nu = 0$, i.e. stock returns are not predictable and uncorrelated with the short rate risk.
- For the case of only predictable stock returns, we set $\phi_r = \phi_y = 0$, i.e. bond returns are not predictable, but still correlated with stock returns.

For these three special cases, we derive the discrete-time VAR, re-estimate the discrete-time coefficients, and back out the corresponding continuous-time parameters.

F Discussion on TAA vs Discrete Rebalancing

We consider a trading strategy closely related with TAA, namely discrete rebalancing. The investor using a discrete rebalancing strategy employs a buy-and-hold strategy during the period between two rebalancing dates. On the other hand, TAA keeps the target portfolio weights by trading continuously. However, TAA is very similar to discrete rebalancing and TAA allows us to have closed-form solutions. This feature is especially useful when an investment opportunity set is a function of more than one state variable, as in our model. Solving optimal weights of discrete rebalancing requires computationally burdensome numerical techniques. Thus, in this section we consider a simpler model to compare TAA and discrete rebalancing with one state variable, the divided yield, which governs the expected return of equities. This implies that the short rate is constant and the term structure is flat. Stock returns now follow

$$
\frac{dS(t)}{S(t)} = (r + \sigma_s \Lambda_s(t))dt + \sigma_s dZ_{S,2}(t).
$$

(64)

Dividend yields follow the same mean-reverting process as equation (14), and the price of risk $\Lambda_s$ takes the same form as equation (15). An investor allocates her wealth in the stock and the risk-free asset, which pays a constant return $r$. 

9
We can derive the optimal portfolio weights and value functions of continuous Merton and TAA policy as we do for the full multivariate model. We now provide a solution method to derive the optimal portfolio weights at each point and to compute the discrete rebalancing value function. Suppose that the agent is allowed to trade only \( n \) times at evenly spaced dates over the investment horizon \( T \). We treat time zero as the zero-th rebalancing date. Define the value function at the \( k \)-th rebalancing date as

\[
J_k(W_k, z_k) = \max_{\{w_i\}_{i=k,\ldots,n-1}} \mathbb{E}_k \left[ \frac{W^{1-\gamma}_T}{1-\gamma} \right],
\]

where \( \mathbb{E}_k \) is the conditional expectation on the information upto \( k \)-th rebalancing date, and \( w_i \) is a portfolio weight in stock at \( i \)-th rebalancing date. Then, the following holds

\[
J_k(W_k, z_k) = \max_{w_k} \mathbb{E}_k \left[ J_{k+1}(W_{k+1}, z_{k+1}) \right].
\]

We conjecture that \( \tilde{J}_k(W_k, z_k) = \left( \frac{W_k^{1-\gamma}}{1-\gamma} \right) \tilde{F}_k(z_k)^\gamma \) with a boundary condition \( \tilde{F}_n(z_n) = 1 \). Plugging this into the above equation, we get

\[
\tilde{F}_k(z_k)^\gamma = \min_{w_k} \mathbb{E}_k \left[ \tilde{F}_{k+1}(z_{k+1})^\gamma \left( e^{r\Delta} + w_k g \right)^{1-\gamma} \right],
\]

where \( \Delta \) is a trading interval and

\[
g = \exp \left( \int_{t_k}^{t_{k+1}} \log S(u)du \right) - e^{r\Delta}.
\]

The first order condition is

\[
\mathbb{E}_k \left[ \tilde{F}_{k+1}(z_{k+1})^\gamma \left( e^{r\Delta} + w_k g \right)^{-\gamma} g \right] = 0,
\]

which we solve by Gaussian Quadrature. By plugging the optimal portfolio weight policy into equation (67), we obtain \( \tilde{F}_k(z_k) \). Doing this recursively, we derive the portfolio weights rule at each rebalancing date, and solve for \( \tilde{F}_0(z_0) \).

To capture an effect of inability to trade in more detail, we compute utility costs of a buy-and-hold strategy versus a TAA strategy over \( T \) periods. The TAA strategy rebalances back to a constant portfolio weight, and thus is a single-switching TAA strategy. The utility costs are reported in percentage terms of initial wealth and represent the increase in initial wealth required to make the buy-and-hold investor have the same utility if she had the ability to undertake a TAA strategy with a single switch. We take the same parameters as Panel C of Table A-2, except we set \( r \) at a constant level \( r = \bar{r} \). We set the risk aversion to be \( 7.9 \), which is in line with the
results in the paper. We compute the utility costs integrating across the steady-state distribution of the single state variable, the dividend yield. Figure A-1 plots the results. As we expect, utility costs of buy-and-hold policy are positive. As the horizon increases, the utility costs increase. However, even in 10-year of horizon the utility cost is less than 0.9%, which indicates that we can take TAA as good approximation for discrete rebalancing.
References


Table A-1: Summary Statistics of State Variables and Asset Returns

<table>
<thead>
<tr>
<th>Correlations</th>
<th>Short Rate</th>
<th>Term Spread</th>
<th>Div. Yield</th>
<th>Excess Bond Ret</th>
<th>Excess Stock Ret</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean (%)</td>
<td>3.86</td>
<td>0.70</td>
<td>3.62</td>
<td>1.35</td>
<td>6.72</td>
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<tr>
<td>Stdev (%)</td>
<td>2.94</td>
<td>0.77</td>
<td>1.54</td>
<td>0.07</td>
<td>14.55</td>
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<tr>
<td>Short Rate</td>
<td>1.00</td>
<td>-0.61</td>
<td>-0.17</td>
<td>0.13</td>
<td>-0.02</td>
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<tr>
<td>Term Spread</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

The table reports means, standard deviations, and correlations of the short rate, term spread, dividend yield, and excess returns of bonds and stocks. We take monthly frequency data from January 1941 to December 2013. The short rate is the three-month T-bill rate and the term spread is the difference between the 10-year and two-year Treasury bond yields. The stock data is the monthly return of S&P 500 index. The bond data is the monthly return of 10-year Treasury constant maturity bond index. All data are continuously compounded and means and standard deviations are annualized.
<table>
<thead>
<tr>
<th></th>
<th>Panel A: IID Returns</th>
<th></th>
<th>Panel B: Predictable Bond Returns Only</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>VAR Parameters</td>
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<td>VAR Parameters</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>K</td>
</tr>
<tr>
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<td>$K$</td>
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<tr>
<td></td>
<td>$r$</td>
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<td>$y$</td>
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<tr>
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<td>Volatility Parameters</td>
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<td></td>
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<td></td>
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<tr>
<td></td>
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<td>$\lambda_s$</td>
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<td>-</td>
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</tr>
</tbody>
</table>

Table A-2: Continuous-Time Parameters
### Panel C: Predictable Stock Returns Only

**VAR Parameters**

\[
\begin{array}{cccc}
\theta & r & y & z \\
r & 0.0387 & 0.1119 & - & - \\
 y & 0.0078 & - & 0.4187 & - \\
 z & 0.0374 & - & - & 0.2121 \\
\end{array}
\]

**Volatility Parameters**

\[
\begin{array}{cccc}
Z_r & Z_y & Z_z & Z_s \\
r & 0.0134 & - & - & - \\
 y & -0.0050 & 0.0054 & - & - \\
 z & - & - & 0.0097 & - \\
 dS/S & 0.0000 & - & -0.0797 & 0.1218 \\
\end{array}
\]

**Prices of Risk**

\[
\begin{array}{cccc}
\phi & \\
\lambda & r & y & z \\
 \Lambda_r & -0.3020 & - & - & - \\
 \Lambda_s & -0.1496 & - & - & 18.893 \\
\end{array}
\]

### Panel D: Predictable Stock and Bond Returns

**VAR Parameters**

\[
\begin{array}{cccc}
\theta & r & y & z \\
r & 0.0358 & 0.1273 & 0.2913 & - \\
 y & 0.0082 & - & 0.2838 & - \\
 z & 0.0375 & - & - & 0.2108 \\
\end{array}
\]

**Volatility Parameters**

\[
\begin{array}{cccc}
Z_r & Z_y & Z_z & Z_s \\
r & 0.0134 & - & - & - \\
 y & -0.0050 & 0.0054 & - & - \\
 z & - & - & 0.0097 & - \\
 dS/S & -0.0061 & - & -0.0798 & 0.1217 \\
\end{array}
\]

**Prices of Risk**

\[
\begin{array}{cccc}
\phi & \\
\lambda & r & y & z \\
 \Lambda_r & -0.0293 & -11.0105 & -21.754 & - \\
 \Lambda_s & -0.1638 & - & - & 18.599 \\
\end{array}
\]
Note to Table A-2
We report continuous-time parameters corresponding to systems with no predictability (Panel A), predictability of bond returns only (Panel B), only predictable stock returns (Panel C), and when expected returns of both assets vary over time (Panel D). Panel D corresponds to the results in Table 1 of the main paper. For the other systems, we re-estimate the model with restricting some coefficients to be zero and recover the corresponding continuous-time parameters.
The figure plots utility costs of a buy-and-hold strategy versus a TAA strategy over $T$ periods. The TAA strategy rebalances back to a constant portfolio weight, and thus is a single-switching TAA strategy. The utility costs are reported in percentage terms of initial wealth and represent the increase in initial wealth required to make the buy-and-hold investor have the same utility if she had the ability to undertake a TAA strategy with a single switch. We take a risk aversion of 7.9 to be in line with the results in the paper. We compute the utility costs integrating across the steady-state distribution of the single state variable, the dividend yield.