Time Series and Dynamic Models

Section 7

Stochastic Volatility Models

(pictures courtesy of Prof. Hedi Lopes – The University of Chicago)

Carlos M. Carvalho

The University of Texas at Austin

SV Model

$$
\begin{aligned}
y_t &= exp\left(\frac{\lambda_t}{2}\right)\epsilon_t \\
\lambda_t &= \alpha + \beta\lambda_{t-1} + w_t
\end{aligned}
$$

- $\epsilon_t \sim N(0,1)$ and $w_t \sim N(0,\omega^2)$
- $y_t \sim N(0, e^{\lambda_t})$
- $|\beta| < 1$... stationary model for the log-volatility

Posterior Inference

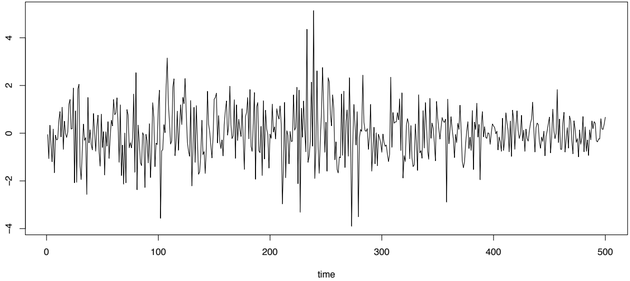
The overal Gibbs Sampler will cycle through the following:

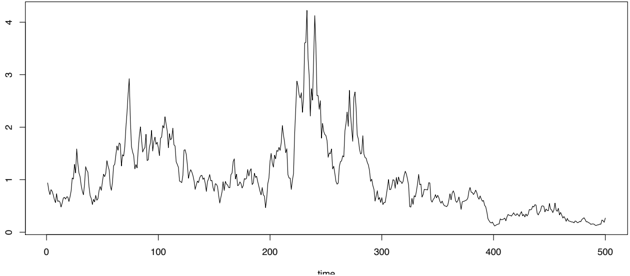
1. $p(\alpha|\beta, \omega^2, \lambda_{1:T}, D_T)$
2. $p(\beta|\alpha, \omega^2, \lambda_{1:T}, D_T)$
3. $p(\omega^2|\alpha, \beta, \lambda_{1:T}, D_T)$
4. $p(\lambda_{1:T}|\alpha, \beta, \omega^2, D_T)$

We will explore a couple of different ways to explore the full conditional posterior distribution of all states...

- Individual state update via RW Metropolis-Hastings
- Individual state update via Independent Metropolis-Hastings
- Block update via normal approximation
- Block update via mixture approximation

Simulated Data: $\alpha = -0.00645$, $\beta = 0.99$ and $\omega^2 = 0.15^2$





RW Metropolis-Hastings

Let $\Theta = (\alpha, \beta, \omega^2)$. The goal here is to generate draws from the full conditional of each individual state... $p(\lambda_t | \lambda_{-t}, D_T, \Theta)$, for all t.

1. Current state: $\lambda_t^{(j)}$
2. draw λ_t^* from the proposal $N(\lambda_t^{(j)}, v^2)$. v^2 is the tuning parameter...
3. Move to the new state according to the transition

$$\lambda_t^{(j+1)} = \begin{cases} \lambda_t^* & \text{w.p.} \quad \alpha^* \\ \lambda_t^{(j)} & \text{w.p.} \quad 1 - \alpha^* \end{cases}$$

where

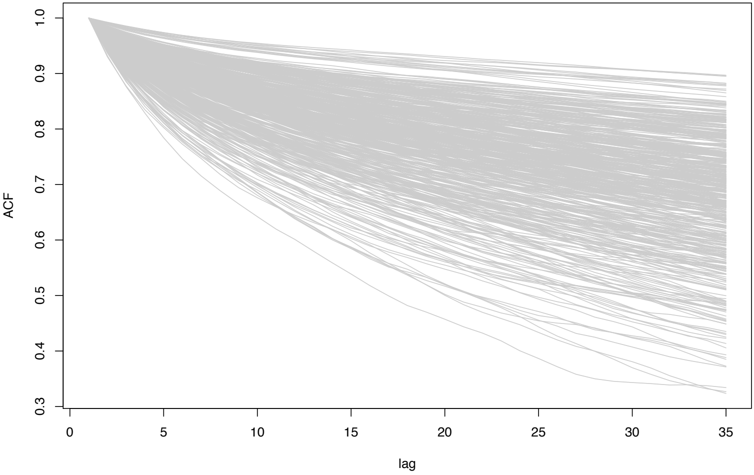
$$\alpha^* = \min \left\{ 1, \frac{p\left(\lambda_t^* | \lambda_{-t}, \Theta, D_T\right)}{p\left(\lambda_t^{(j)} | \lambda_{-t}, \Theta, D_T\right)} \right\}$$

RW Metropolis-Hastings

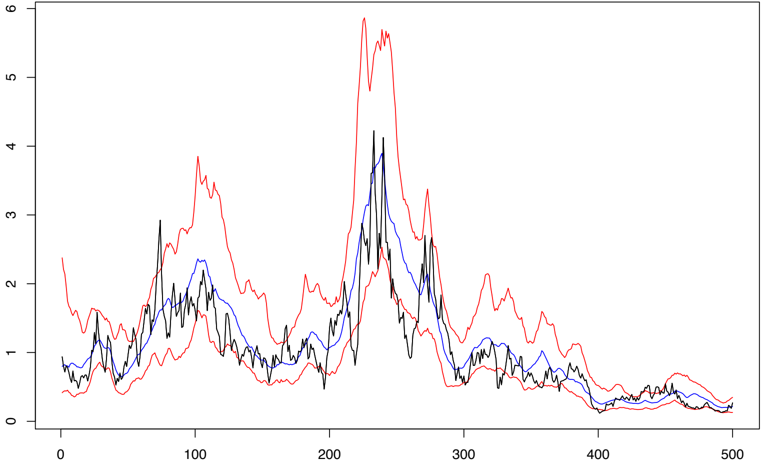
As a reminder,

$$p(\lambda_t|\lambda_{-t},\Theta,D_T) \propto p(y_t|\lambda_t)p(\lambda_t|\lambda_{t-1},\Theta)p(\lambda_{t+1}|\lambda_t,\Theta)$$

RW M-H



RW M-H



Independence M-H

Let's start by trying to figure out a proposal distribution... Notice that we can re-write

$$p(\lambda_t|\lambda_{-t}, D_T, \Theta) \propto p(\lambda_t|\lambda_{t-1}, \lambda_{t+1}, \Theta)p(y_t|\lambda_t)$$

Here is an idea for a proposal:

$$q(\lambda_t|\lambda_{-t}, D_T, \Theta) = N(\hat{\mu}_t, \nu_t^2)$$

where $\hat{\mu}_t = \mu_t + 0.5\nu_t^2(y_t^2 e^{-\mu_t} - 1)$

and
$\mu_t = \mathrm{E}(\lambda_t|\lambda_{t-1}, \lambda_{t+1}, \Theta)$
$\nu_t^2 = \mathrm{Var}(\lambda_t|\lambda_{t-1}, \lambda_{t+1}, \Theta)$

Independence M-H

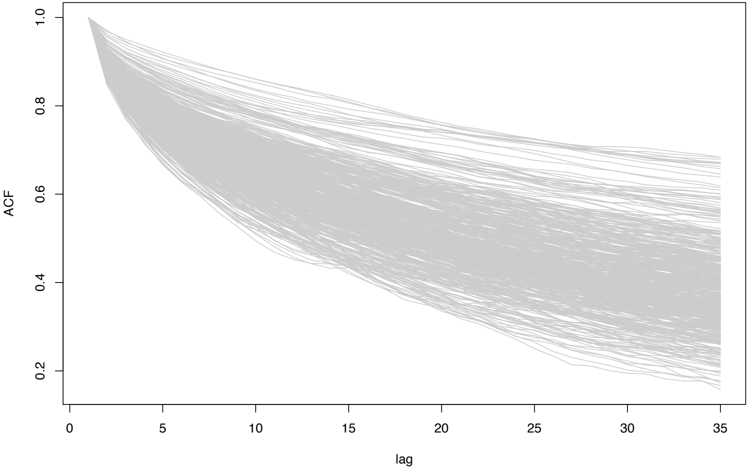
1. Current state: $\lambda_t^{(j)}$
2. draw λ_t^* from the proposal $q(\lambda_t) = N(\hat{\mu}_t, \nu_t^2)$
3. Move to the new state according to the transition

$$\lambda_t^{(j+1)} = \begin{cases} \lambda_t^* & \text{w.p.} \quad \alpha^* \\ \lambda_t^{(j)} & \text{w.p.} \quad 1-\alpha^* \end{cases}$$

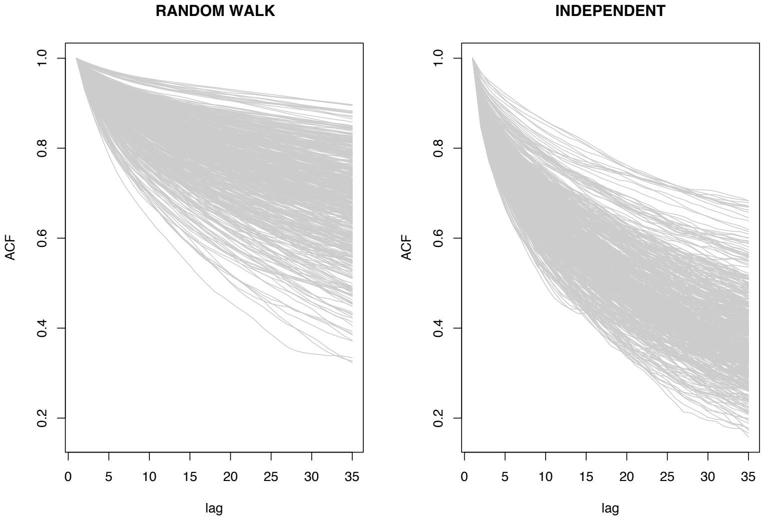
where

$$\alpha^* = \min\left\{1, \frac{p\left(\lambda_t^* | \lambda_{-t}, \Theta, D_T\right)}{p\left(\lambda_t^{(j)} | \lambda_{-t}, \Theta, D_T\right)} \times \frac{q(\lambda_t^{(j)})}{q(\lambda_t^*)}\right\}$$

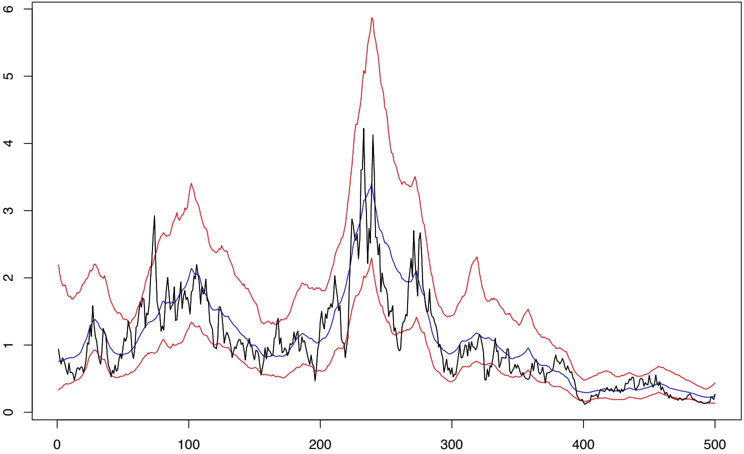
Independence M-H



Independence M-H



Independence M-H



Normal Approximation

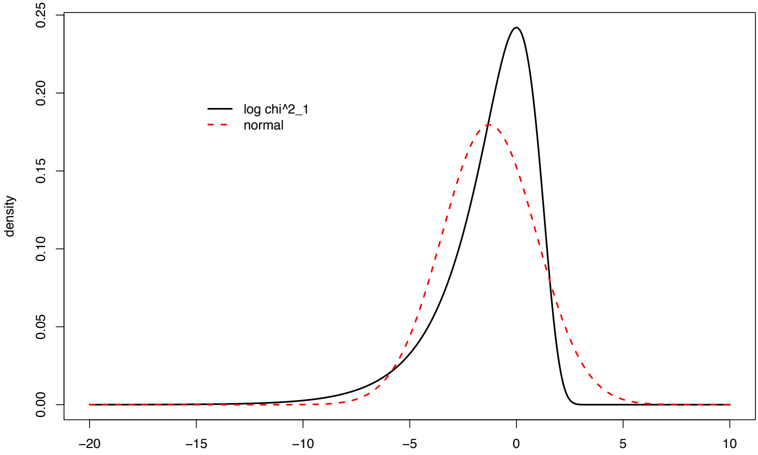
Another alternative is to use the following approximation for the model...
Let $y_t^* = \log(y_t^2)$ and $\eta_t = \log(\epsilon_t^2)$. The model can be re-written as:

$$\begin{aligned} y_t^* &= \lambda_t + \eta_t \\ \lambda_t &= \alpha + \beta\lambda_{t-1} + w_t \end{aligned}$$

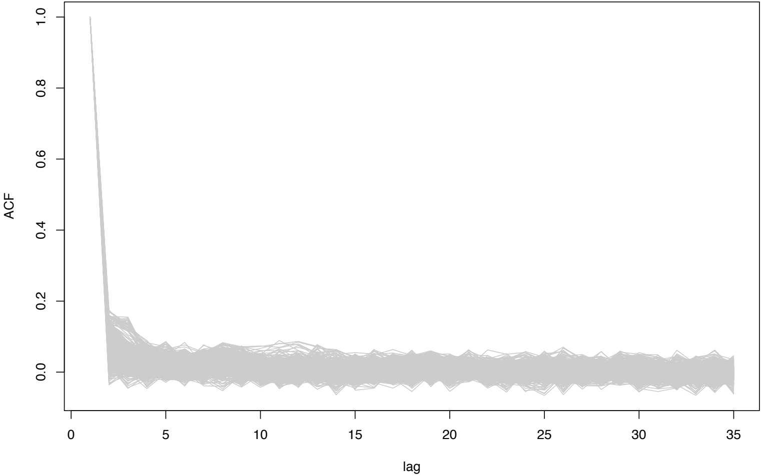
If η_t was normally distributed, life would be easy, right? However, $\eta_t \sim \log(\chi_1^2)$ with $\mathrm{E}(\eta_t) = -1.27$ and $\mathrm{Var}(\eta_t) = 4.935$

Well, how about using the approximation $\eta_t \sim N(-1.27, 4.935)$? That would allow us to draw the states in block using the FFBS algorithm!!

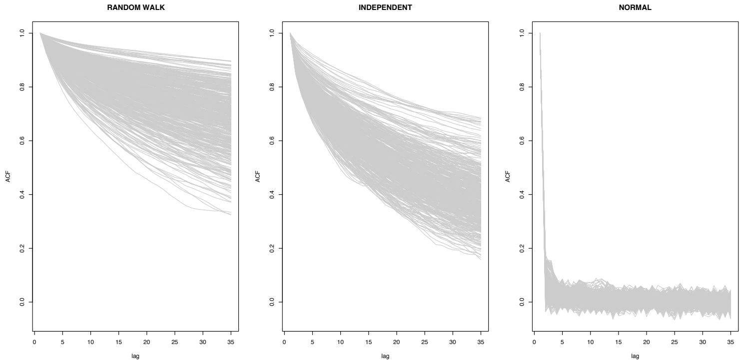
Normal Approximation



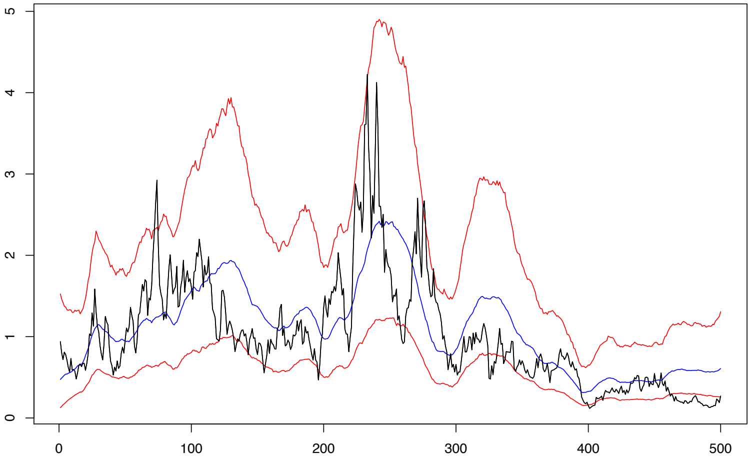
Normal Approximation



Normal Approximation



Normal Approximation



Mixture Approximation

The $\log \chi_1^2$ distribution can be approximated by

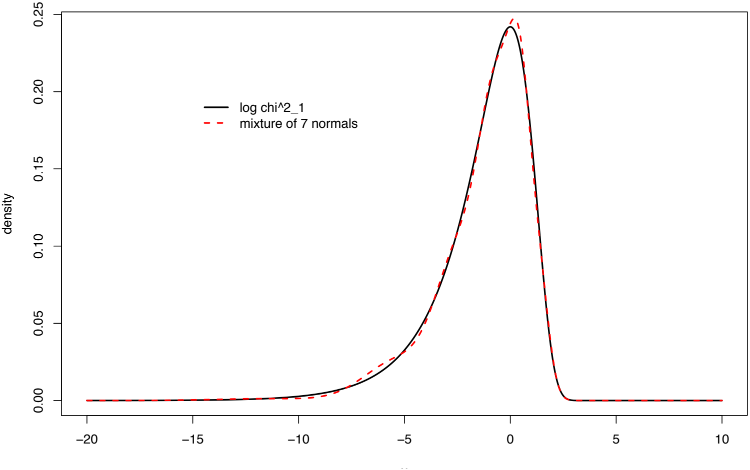
$$\sum_{i=1}^{7} \pi_i N(\mu_i, \omega_i^2)$$

where

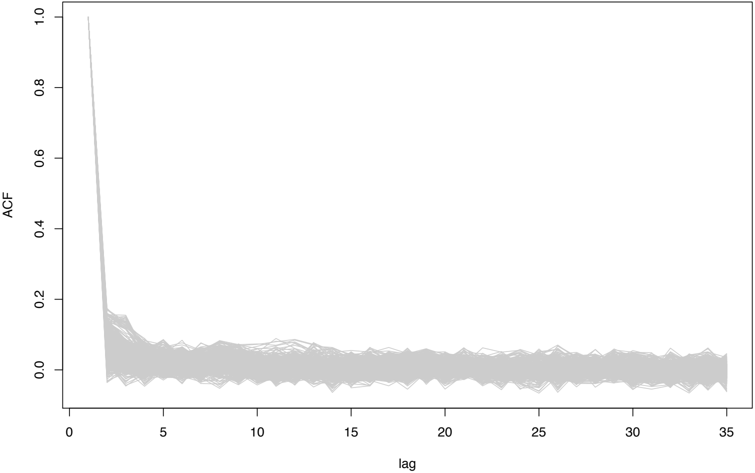
i	π_i	μ_i	ω_i^2
1	0.00730	-11.40039	5.79596
2	0.10556	-5.24321	2.61369
3	0.00002	-9.83726	5.17950
4	0.04395	1.50746	0.16735
5	0.34001	-0.65098	0.64009
6	0.24566	0.52478	0.34023
7	0.25750	-2.35859	1.26261

This approximation also allow us to use the FFBS algorithm... (next class)

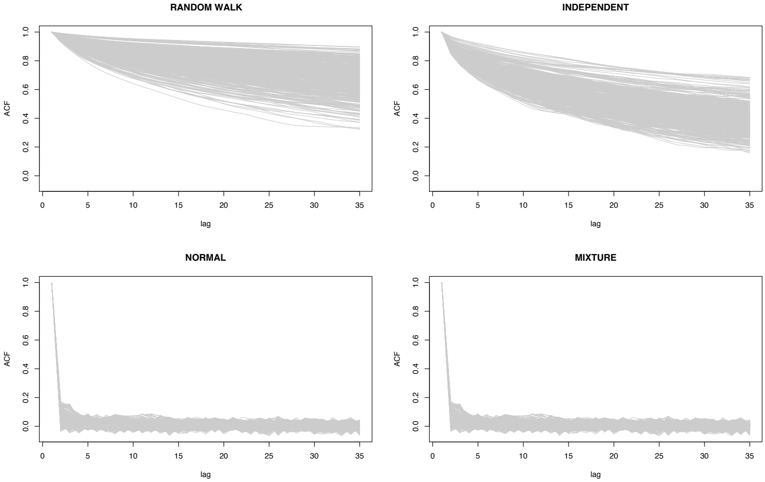
Mixture Approximation



Mixture Approximation



Mixture Approximation



Mixture Approximation