Time Series and Dynamic Models

Section 6
Dynamic Regression Model

Carlos M. Carvalho
The University of Texas at Austin

Normal/Gamma

Quick review...

Let $\phi \sim Ga\left(\frac{n_0}{2}, \frac{d_0}{2}\right)$ and $(X|\phi) \sim N(m, C\phi^{-1})$
hence $(\phi|X) \sim Ga\left(\frac{n_1}{2}, \frac{d_1}{2}\right)$ where

$$n_1 = n_0 + 1 \quad \text{and} \quad d_1 = d_0 + \frac{(X-m)^2}{C}$$

Also,

$$p(X) \propto [n_0 + (X-m)^2/R]^{-\frac{n+1}{2}}$$

so that $X \sim T_{n_0}(m, R)$, where $R = C\left(\frac{d_0}{n_0}\right)$

Dynamic Regression Model

The Model:

$$\begin{aligned} y_t &= \beta_t X_t + \epsilon_t \\ \beta_t &= \beta_{t-1} + w_t \end{aligned}$$

- $\epsilon_t \sim N(0, \sigma^2)$ and $w_t \sim T_{n_{t-1}}(0, W_t)$
- $W_t = C_{t-1} \left(\frac{1-\delta}{\delta}\right)$
- δ is the discount factor $\in (0, 1)$

Posterior at $t-1$:

$$(\beta_{t-1}|D_{t-1}) \sim T_{n_{t-1}}(m_{t-1}, C_{t-1}) \qquad (\sigma^{-2}|D_{t-1}) \sim Ga\left(\frac{n_{t-1}}{2}, \frac{d_{t-1}}{2}\right)$$

Dynamic Regression Model

Priors at t :

$$(\beta_t|D_{t-1}) \sim T_{n_{t-1}}(m_{t-1}, R_t) \qquad (y_t|D_{t-1}) \sim T_{n_{t-1}}(f_t, Q_t)$$

where
$R_t = C_{t-1}/\delta$, $f_t = m_{t-1}X_t$, $Q_t = X_t^2 R_t + S_{t-1}$ and $S_{t-1} = \frac{d_{t-1}}{n_{t-1}}$

Posteriors at t :

$$(\beta_t|D_t) \sim T_{n_t}(m_t, C_t) \qquad (\sigma^{-2}|D_t) \sim Ga\left(\frac{n_t}{2}, \frac{d_t}{2}\right)$$

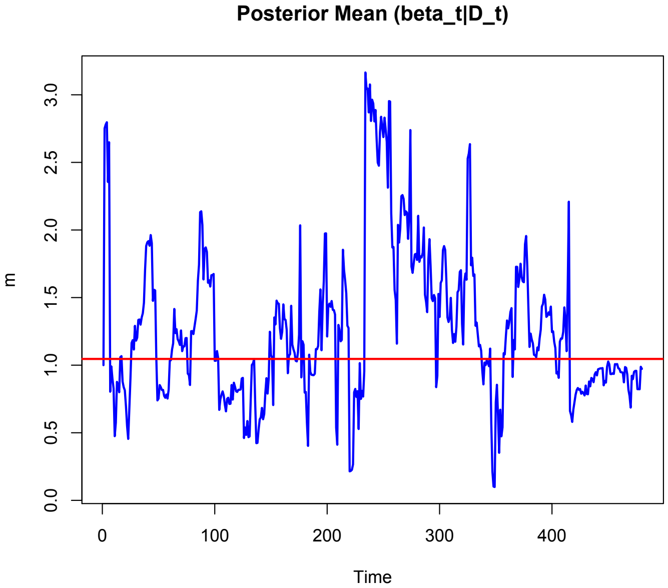
where $n_t = n_{t-1} + 1$, $d_t = d_{t-1} + S_{t-1}e_t^2/Q_t$,

$$m_t = m_{t-1} + A_t e_t \qquad C_t = R_t S_t / Q_t$$

$e_t = (y_t - f_t)$, $A_t = X_t R_t / Q_t$ and $S_t = \frac{d_t}{n_t}$

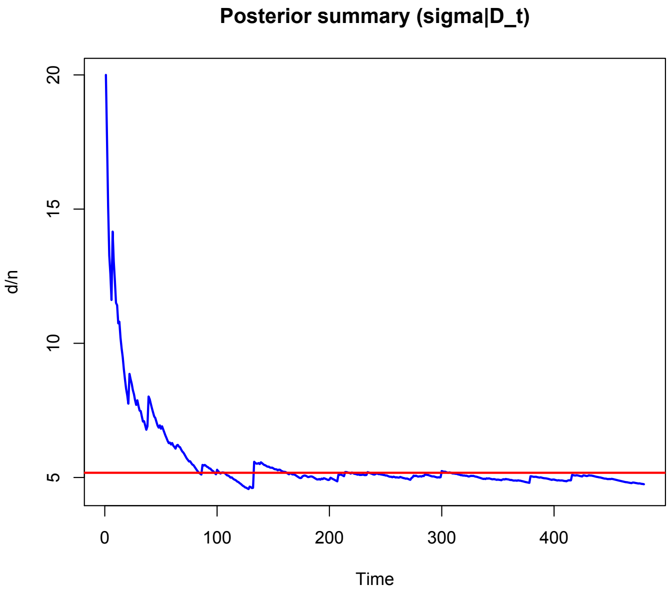
Example: APPL vs. MKT

$\delta = 0.90$



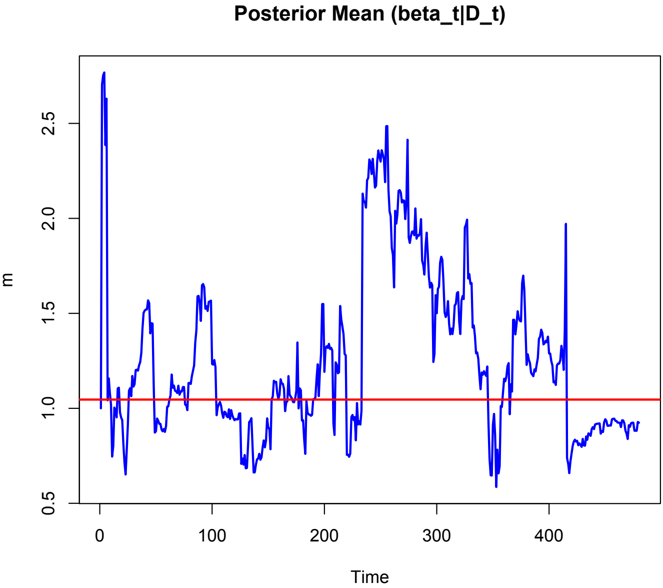
Example: APPL vs. MKT

$\delta = 0.90$



Example: APPL vs. MKT

$\delta = 0.95$



Example: APPL vs. MKT

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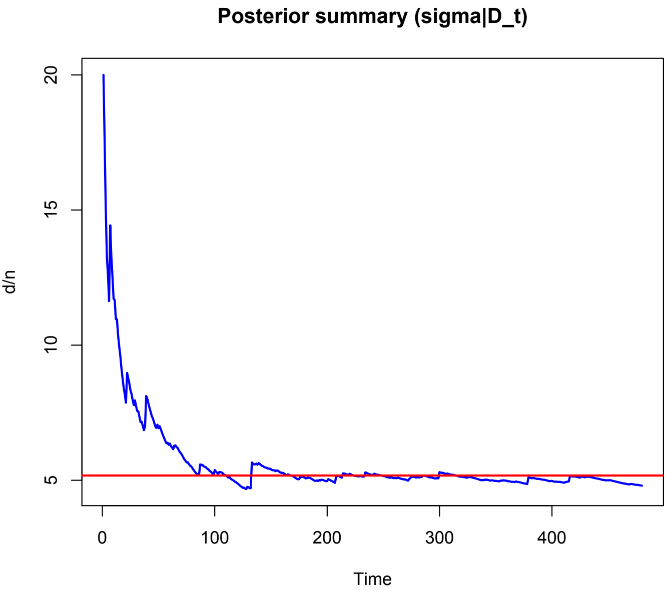
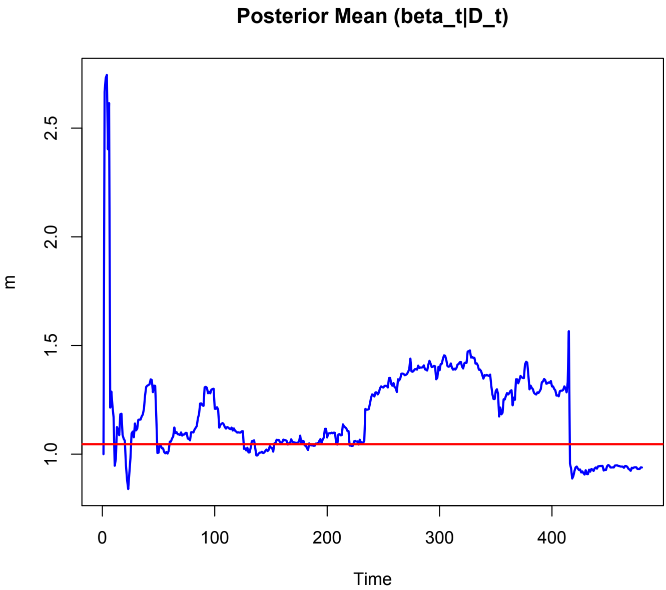


Example: APPL vs. MKT

$\delta = 0.99$



Example: APPL vs. MKT

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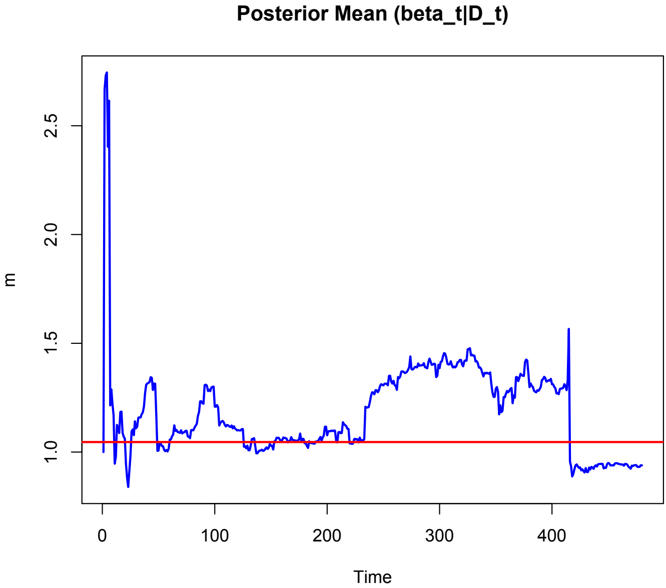
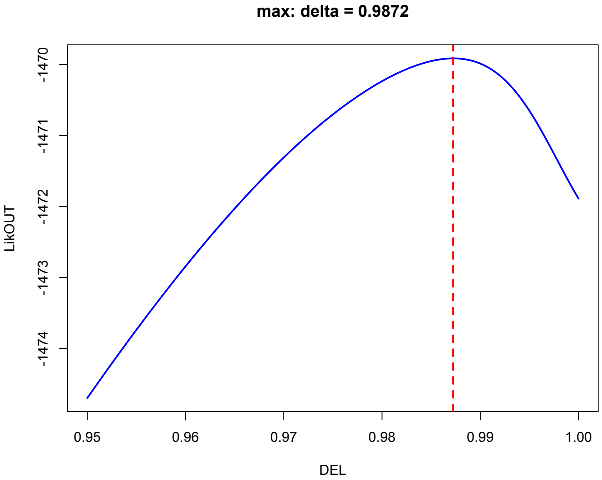


Example: APPL vs. MKT

We can compute the “model likelihood” for different values of δ...

$$p(\delta_i | D_t) \propto p(y_{1:T} | \delta_i) = \prod_{t=1}^{T} p(y_t | D_{t-1}, \delta_i)$$



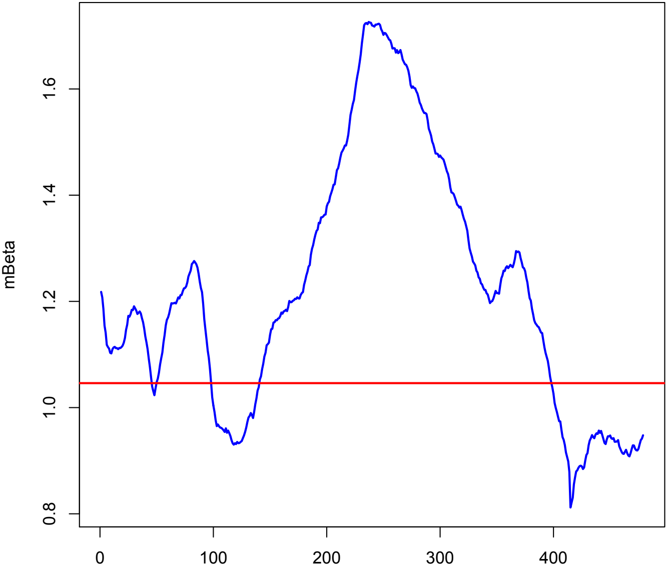
Now let's try the MCMC

The Model:

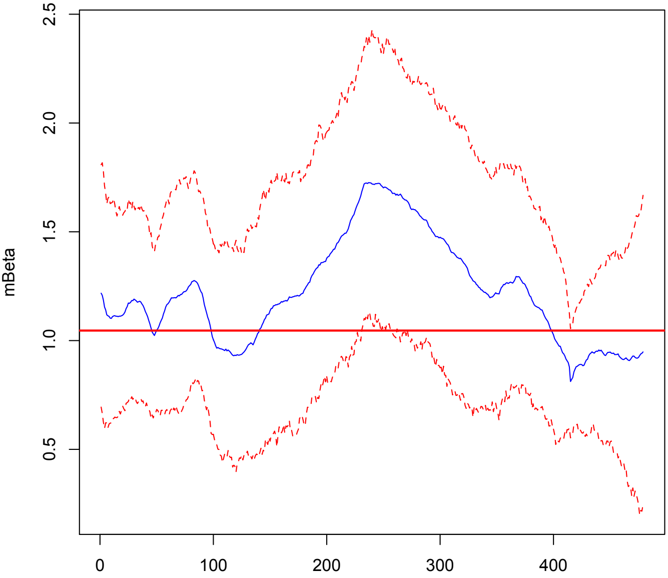
$$\begin{aligned} y_t &= \beta_t X_t + \epsilon_t \\ \beta_t &= \beta_{t-1} + w_t \end{aligned}$$

- $\epsilon_t \sim N(0, \sigma^2)$ and $w_t \sim N(0, \omega^2)$
- $\sigma^{-2} \sim Ga(\frac{a_\sigma}{2}, \frac{b_\sigma}{2})$
- $\omega^{-2} \sim Ga(\frac{a_\omega}{2}, \frac{b_\omega}{2})$

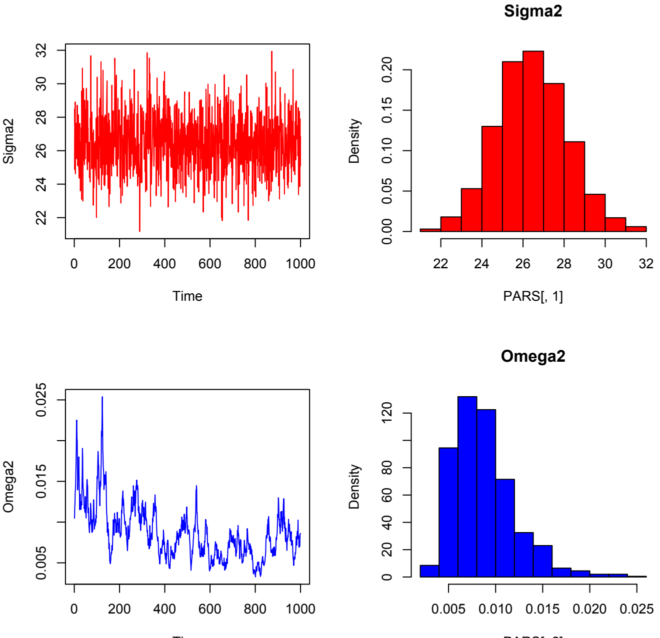
Example: APPL vs. MKT



Example: APPL vs. MKT



Example: APPL vs. MKT



Homework

1. Replicate the dynamic regression example from section 6.
    - First make sure you can implement the closed-form solution using discount factors
    - evaluate the the model likelihood for different values of δ
2. Simulate, for different values of r, data from the AR(1) plus noise model with $\alpha = 0$ and $\beta = 1$. $r = W/V$ the ratio between the variance in the evolution equation and the variance in the observation equation. For the different choices of r plot the time series generated and the smoothed posterior mean from the distribution of $\theta_t | D_T$.
3. Prove the result in slide 6 in Section 5.