On the Long Run Volatility of Stocks

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\(^1\) with Hedibert Lopes and Rob McCulloch.
Question

- What is the long-run variance of stock returns?
Stocks for the Long Run: Conventional Wisdom

**Figure 1-4**

Total Real Return Indexes, 1802 through December 2006

- **Stocks**
  - $755,163

- **Bonds**
  - $1,083

- **Bills**
  - $301

- **Gold**
  - $1.95

- **Dollar**
  - $0.06

Stocks for the Long Run: Conventional Wisdom

▶ x-axis: Horizon     y-axis: Volatility per year

Sample Volatility per Year: 1802-2009
Figure 6. Predictive variance of multiperiod return and its components. Panel A plots the variance of the predictive distribution of long-horizon returns, $\text{Var}(r_{T,T+k}|D_T)$. Panel B plots the five components of the predictive variance. All quantities are divided by $k$, the number of periods in the return horizon. The results are obtained by estimating the predictive system on annual real U.S. stock market returns in 1802 to 2007. Three predictors are used: the dividend yield, the bond yield, and the term spread.
Summary

Taking a conditional approach, from the investor’s perspective:

- A simple view of the world suggests that stocks are less volatile over long horizons (Barberis, 2000)...

- …while a more complex view of the world states that stocks could be more volatile over long horizons (Pastor and Stambaugh, 2012)
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Our work hopes to address:

1. Which direction is right?
2. Better understand the results sensitivity to prior specification
3. Enrich PS2012 framework to include time-varying volatilities

Our results indicates that I am not crazy for having 100% equity in my retirement portfolio, i.e., stocks are indeed less volatile in the long-run.
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Returns $k$ periods in the future:

$$r_{1,k} = r_1 + r_2 + r_3 + \cdots + r_k$$
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If returns are i.i.d. $r_i \sim N(\mu, \sigma^2)$, i.e., $\approx$ random walk on prices:

$$Var(r_{1,k}) = k\sigma^2$$

so that the variance per period is constant for any $k$ investment horizon.
However, investors face parameter uncertainty...

If \( r_t \sim N(\mu, \sigma^2) \) and \( \mu \) is unknown, then,

\[
\text{Var}(r_{t,t+k}|D_t) = E\{\text{Var}(r_{t,t+k}|\mu, D_t)\} + \text{Var}\{E(r_{t,t+k}|\mu, D_t)\}
\]

\[
= k\sigma^2 + k^2 \text{Var}(\mu|D_t)
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$$= k\sigma^2 + k^2 \text{Var}(\mu|D_t)$$

Long run volatility (predictive variance) grows linearly with the horizon
Background

If $\mu$ is mean reverting and

\[
\begin{align*}
    r_{t+1} &= \mu_t + u_{t+1} \\
    \mu_{t+1} &= \alpha + \beta \mu_t + w_{t+1}
\end{align*}
\]

where $\text{Corr}(u_{t+1}, w_{t+1}) < 0$,

\[
\text{Var}(r_{t,t+k}) = k\sigma^2 [1 + 2A\rho_{uw} + B]
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$$

The effect of $\rho_{uw}$ effect can dominate and imply a decreasing long run risk as both $A > 0$ and $B > 0$. 

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For stocks... using dividend yield as a proxy for expected returns:

\[
\rho = -0.903
\]
The General Model

\[ r_{t+1} = \mu_t + u_{t+1} \]
\[ \mu_{t+1} = \alpha + \beta \mu_t + w_{t+1} \]
\[ x_{t+1} = A + Bx_t + v_{t+1} \]

where

\[
\begin{pmatrix}
  u_{t+1} \\
  w_{t+1} \\
  v_{t+1}
\end{pmatrix}
\sim \mathcal{N}(0, \Sigma_{t+1})
\]
Priors and Posteriors... “weak prior” on $\rho$
Priors and Posteriors... “strong prior” on $\rho$
“Weak” vs. “Strong” Prior on $\rho$

**Figure:** Histograms of draws from priors (gray) and posteriors (red) for $\rho_{uw}$ in the “weak prior” (left) and “strong prior” (right) specifications.
Prior and Posterior Predictives

![Graph showing prior and posterior predictive distributions. The x-axis represents horizon in years, ranging from 0 to 50. The y-axis represents vol per period (% per year), ranging from 0 to 2.0. The graph includes two lines: a dashed line for the prior predictive and a solid line for the posterior predictive. The prior predictive shows a steep increase as the horizon increases, while the posterior predictive remains relatively flat.]
Long Run Volatilities per Period

![Graph showing long run volatilities per period](image)

- iid
- rho=0
- weak prior
- strong prior
- time-varying Sigma
- unconditional

**Axes:**
- Y-axis: Vol per Period (% per year)
- X-axis: Horizon (years)
How Robust is the Result?

Figure: Histograms of draws from priors (gray) and posteriors (red) for $\rho_{uw}$ in the “weak prior” set up where the 207 observations have been reshuffled so to break its dynamics.
Sources of Variance

Figure: Decomposition of the predictive variance per period. The left panel is the results from the “weak prior” set up while the “strong prior” is in the right panel.
Figure: Predictive volatility per period plotted for different horizons for fixed values of $\beta$ in the “weak prior” set up.
Figure: Left panel shows the posterior distribution of $\mu_{T+30}$. The solid line in the right panel results from the “weak” prior set-up while the dashed line fixes $\beta = 0.945$. The right panel shows the decomposition of the predictive variance per period for the case where $\beta = 0.945$. 
Portfolio Implications

The graph illustrates the percentage stock allocation as a function of horizon (years) for different scenarios:

- **Strong Prior**: A solid blue line showing an upward trend.
- **Fixed beta (0.945)**: A solid black dashed line indicating a downward trend.
- **iid**: A dotted black line also showing a downward trend.

As the horizon increases, the percentage stock allocation for the Strong Prior scenario increases, while the allocations for Fixed beta (0.945) and iid decrease.
Replicating PS2012

Figure: Priors (gray) and posteriors (red) draws of $\rho_{uw}$ using priors from Pastor and Stambaugh 2012. In their terminology, from left to right: “non-informative”, “less informative” and “more informative”.
Figure: Comparison of our results (right) to the results using the priors in Pastor and Stambaugh 2012 (left).
Adding Predictors

**Figure:** Predictive volatility per period plotted for different horizons when predictors are added. Results are for the “weak prior” (left) and “strong prior” set up.
Time-Varying Volatilities
Closing Comments:

- With reasonable priors (or maybe even unreasonable) and 200 years of data stocks look very attractive for long horizon portfolios.
- This result appears robust to the added complexity of time varying relationships between predictors and expected returns and stochastic volatility.
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- With reasonable priors (or maybe even unreasonable) and 200 years of data stocks look very attractive for long horizon portfolios.
- This result appears robust to the added complexity of time varying relationships between predictors and expected returns and stochastic volatility.
- The take home message is that conventional wisdom might not be so wrong after all...
Instead of just $\Sigma$, we want $\Sigma_t$, and we want to easily incorporate the prior belief that

$$\rho_t = corr(u_t, w_t) < 0, \text{ for all } t$$

and possibly other prior beliefs as well.
Multivariate Stochastic Volatility

We start with the Choleski Stochastic Volatility approach of Lopes, McCulloch, and Tsay.

With one $\times$ we have:

\[

t = \exp(\theta_t/2) Z_t \\
u_t = \theta_t w_t + \exp(\theta_t/2) Z_t \\
v_t = \phi_t w_t + \phi_t u_t + \exp(\phi_t/2) Z_t
\]

At each $t$, the three $\theta$’s and three $\phi$’s are one to one with $\Sigma_t$.

Let’s just focus on the $\theta$’s because they determine $\rho_t$. 
Multivariate Stochastic Volatility

We have,

\[ w_t = \exp(\theta_t/2) Z_{t1} \]
\[ u_t = \theta_{t3} w_t + \exp(\theta_{t2}/2) Z_{t2} \]

\[ \rho_t = \rho(\theta_{t1}, \theta_{t2}, \theta_{t3}) = \frac{\theta_{t3} \exp(\theta_{t1})}{\left[ \theta_{t3}^2 \exp(\theta_{t1}) \times \exp(\theta_{t1}) \right]^{1/2}} \]
The usual prior for the $\theta_{ti}$ series is

$$\theta_{ti} = a_i + b_i \theta_{t-1,i} + s_i z_{ti}$$

Let’s call this $q(\theta_{ti} \mid \theta_{t-1,i})$.

Letting $\theta_t = (\theta_{t1}, \theta_{t2}, \theta_{t3})$, let,

$$q(\theta_t \mid \theta_{t-1}) = \prod_{i=1}^{3} q(\theta_{ti} \mid \theta_{t-1,i}).$$

We usually choose the $s_i$ so that successive $\theta$ are not “too different”.
Prior Formulation

Our prior formulation is

\[ p(\theta_t | \theta_{t-1}) \propto q(\theta_t | \theta_{t-1}) f(\theta_t). \]
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\[ f(\theta_t) = \exp \left\{ -\frac{(\rho(\theta_t) - \bar{\rho})^2}{\kappa} \right\} \]
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q:
usual smoothness, don’t let \( \theta \)'s jump around too much

f:
have preference for each \( \theta_t \), small \( \kappa \) means each \( \theta_t \) should be such that \( \rho_t \approx \bar{\rho} \)
Bivariate Stochastic Volatility with Flexible Prior

\[(w_t, u_t)' \sim N(0, \Sigma(\theta_t)), \quad \theta_t = (\theta_{t1}, \theta_{t2}, \theta_{t3})\]

\[w_t = \exp(\theta_{t1}/2) Z_{t1}\]

\[u_t = \theta_{t3} w_t + \exp(\theta_{t2}/2) Z_{t2}\]

\[p(\theta_t | \theta_{t-1}) \propto q(\theta_t | \theta_{t-1}) f(\theta_t) = q(\theta_t | \theta_{t-1}) f(\theta_t) K(\theta_{t-1})\]

\[p(\theta_0) \propto f(\theta_0) \Pi_{i=1}^{3} p(\theta_i 0)\]
Let $w$ and $u$ be the observed bivariate series consisting of daily returns from two stocks in the S&P100.
Prior:

$$f(\theta_t) = \exp \left[\frac{-(\rho(\theta_t) - \bar{\rho})^2}{\kappa}\right]$$

For this data, it is more reasonable to believe that $\rho_t > 0$!

I’ll hide the details about $q$ and show results for

$$\bar{\rho} = .8, \quad \kappa = .01, .25$$

$\kappa = .01$: tight prior.

$\kappa = .25$: loose prior.
loose prior: draws from prior

black is average draw, others are individual draws
(1,1): $\rho_t$, (1,2): $\theta_{t1}$
(2,1): $\theta_{t2}$, (2,2): $\theta_{t3}$
loose prior: draws from posterior

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