Internet Appendix for "A Dynamic Model of Characteristic-Based Return Predictability"

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This appendix provides supplementary material to the paper "A Dynamic Model of Characteristic-Based Return Predictability" and is organized as follows. Section I contains the derivations of firm valuations and returns for the model presented in the paper. Section II describes the empirical data sample and presents summary statistics and stylized facts.

I. Derivations of Firm Valuations and Returns

In this appendix we provide details on the derivations of the firm value and return equations. As shown in equation (10) in the paper, the value of firm *i* equals the discounted value of its expected cash flows conditional on all available information:

$$V_t^i = E_t\left[\int_{u=t}^{\infty} e^{-r(u-t)}\left[f_u^i du + \left(\alpha\lambda K_u^i - k_z\right)\left(du + dM_u\right)\right]\right]. \qquad \text{(IA.1)}$$

Using equation (7) in the paper, we write

$$dM_u = \hat{\mu}_u du + \sigma_M d\bar{\omega}_u. \qquad \text{(IA.2)}$$

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Substituting in equation (IA.1), taking the expectation with respect to the Brownian term $d\bar{\omega}_u$, and writing firm value as a function of the state variables, we have

$$V\left(z_t^i, K_t^i, f_t^i, \hat{\mu}_t\right) \equiv E_t\left[\int_{u=t}^{\infty} e^{-r(u-t)}\left[f_u^i + \left(\alpha\lambda K_u^i - k_z\right)\left(1+\hat{\mu}_u\right)\right]du\right]. \tag{IA.3}$$

The first state variable, z_t^i, is the firm's idiosyncratic growth state, which follows the Markov process described in Section I.A of the paper. The laws of motion of the other three state variables are

$$dK_t^i = \left(k_z - \lambda K_t^i\right)\left(\left(1+\hat{\mu}_t\right)dt + \sigma_M d\bar{\omega}_t\right), \tag{IA.4}$$

$$df_t^i = \left(a_z k_z - \lambda f_t^i\right)\left(\left(1+\hat{\mu}_t\right)dt + \sigma_M d\bar{\omega}_t\right), \tag{IA.5}$$

$$d\hat{\mu}_t = -\rho_\mu \hat{\mu}_t dt + \sigma_\mu \eta ds_t + \frac{\gamma}{\sigma_M} d\bar{\omega}_t. \tag{IA.6}$$

Equations (IA.4) and (IA.5) follow from substituting equation (IA.2) in equations (1) and (2) in the paper, respectively. Equation (IA.6) restates equation (6) in the paper.

Using Itô's Lemma, the instantaneous rate of return of the firm, dr_t^i, is given by

$$
\begin{aligned}
dr_t^i V_z = {} & f_t^i dt + \left(\alpha\lambda K_t^i - k_z\right)\left(\left(1+\hat{\mu}_t\right)dt + \sigma_M d\bar{\varpi}_t\right) + dQ_{z,z+1}\left(V_{z+1} - V_z\right) \\
& + \left(k_z - \lambda K_t^i\right)\left(\left(1+\hat{\mu}_t\right)dt + \sigma_M d\bar{\varpi}_t\right)\frac{\partial V_z}{\partial K} + \left(a_z k_z - \lambda f_t^i\right)\left(\left(1+\hat{\mu}_t\right)dt + \sigma_M d\bar{\varpi}_t\right)\frac{\partial V_z}{\partial f} \\
& + \left(-\rho_\mu \hat{\mu}_t dt + \sigma_\mu \eta ds_t + \frac{\gamma}{\sigma_M} d\bar{\varpi}_t\right)\frac{\partial V_z}{\partial \mu} \qquad \text{(IA.7)} \\
& + \frac{\left(k_z - \lambda K_t^i\right)^2 \sigma_M^2}{2}\frac{\partial^2 V_z}{\partial K^2} + \frac{\left(a_z k_z - \lambda f_t^i\right)^2 \sigma_M^2}{2}\frac{\partial^2 V_z}{\partial f^2} + \left(\frac{\sigma_\mu^2 \eta^2}{2} + \frac{\gamma^2}{2\sigma_M^2}\right)\frac{\partial^2 V_z}{\partial \mu^2} \\
& + \left(k_z - \lambda K_t^i\right)\left(a_z k_z - \lambda f_t^i\right)\sigma_M^2 \frac{\partial^2 V_z}{\partial K \partial f} + \left(k_z - \lambda K_t^i\right)\gamma \frac{\partial^2 V_z}{\partial K \partial \mu} + \left(a_z k_z - \lambda f_t^i\right)\gamma \frac{\partial^2 V_z}{\partial f \partial \mu}.
\end{aligned}
$$

In equation (IA.7), the term V_z gives the firm's value when the firm is in growth state $z \in \{EG, MG, NG\}$. Note that this notation suppresses the state variables $\{K_t^i, f_t^i, \hat{\mu}_t\}$. With slight abuse of notation we write V_{z+1} to indicate the value of the firm in the next growth state the firm will transition into (e.g., $z + 1 = MG$ when $z = EG$). Similarly, we use the notation $dQ_{z,z+1}$ for the Poisson process governing the growth state, and below we use $q_{z,z+1}$ to denote the transition rate (e.g., $q_{z,z+1} = q_{EG}$ when $z = EG$). When the firm is in the no-growth state $z = NG$, V_{z+1} corresponds to the firm's liquidation value (i.e., the sum of the present value of active projects' cash flows and the value expected from the recovery of capital), which we characterize below.

Computing the expected value of the right-hand side of equation (IA.7) and using the fact that the firm's expected return equals the discount rate r, we obtain the partial differential equation that characterizes the firm's value:

$$
\begin{aligned}
rV_z =\ & f_t^i + \left(\alpha\lambda K_t^i - k_z\right)\left(1+\hat{\mu}_t\right) + q_{z,z+1}\left(V_{z+1} - V_z\right) \\
& + \left(k_z - \lambda K_t^i\right)\left(1+\hat{\mu}_t\right)\frac{\partial V_z}{\partial K} + \left(a_z k_z - \lambda f_t^i\right)\left(1+\hat{\mu}_t\right)\frac{\partial V_z}{\partial f} - \rho_\mu \hat{\mu}_t \frac{\partial V_z}{\partial \mu} \\
& + \frac{\left(k_z - \lambda K_t^i\right)^2 \sigma_M^2}{2}\frac{\partial^2 V_z}{\partial K^2} + \frac{\left(a_z k_z - \lambda f_t^i\right)^2 \sigma_M^2}{2}\frac{\partial^2 V_z}{\partial f^2} + \frac{1}{2}\left(\sigma_\mu^2 \eta^2 + \frac{\gamma^2}{\sigma_M^2}\right)\frac{\partial^2 V_z}{\partial \mu^2} \\
& + \left(k_z - \lambda K_t^i\right)\left(a_z k_z - \lambda f_t^i\right)\sigma_M^2 \frac{\partial^2 V_z}{\partial K \partial f} + \left(k_z - \lambda K_t^i\right)\gamma \frac{\partial^2 V_z}{\partial K \partial \mu} + \left(a_z k_z - \lambda f_t^i\right)\gamma \frac{\partial^2 V_z}{\partial f \partial \mu}.
\end{aligned}
\tag{IA.8}
$$

Equation (12) in the paper results from subtracting equation (IA.8) (with the *dt* terms included) from equation (IA.7), rearranging terms, and defining the firm's idiosyncratic return as

$$
d\varepsilon_t^i = \left(dQ_{z,z+1} - q_{z,z+1}dt\right)\frac{V_{z+1} - V_z}{V_z}. \tag{IA.9}
$$

We now conjecture that firm value in state *z* is separable, with the following functional form:

$$
V_z = f_t^i V_f\left(\hat{\mu}_t\right) + K_t^i V_K\left(\hat{\mu}_t\right) + V_{g,z}\left(\hat{\mu}_t\right). \tag{IA.10}
$$

Substituting the conjectured functional form into equation (IA.8), we obtain

$$
\left(r + \lambda\left(1+\hat{\mu}_t\right)\right)V_f + \left(\gamma\lambda + \rho_\mu \hat{\mu}_t\right)V_f' - \frac{1}{2}\left(\sigma_\mu^2 \eta^2 + \frac{\gamma^2}{\sigma_M^2}\right)V_f'' = 1, \tag{IA.11}
$$

$$
\left(r + \lambda\left(1+\hat{\mu}_t\right)\right)V_K + \left(\gamma\lambda + \rho_\mu \hat{\mu}_t\right)V_K' - \frac{1}{2}\left(\sigma_\mu^2 \eta^2 + \frac{\gamma^2}{\sigma_M^2}\right)V_K'' = \alpha\lambda\left(1+\hat{\mu}_t\right), \tag{IA.12}
$$

$$rV_{g,z} + \rho_\mu \hat{\mu}_t V'_{g,z} - \frac{1}{2}\left(\sigma_\mu^2 \eta^2 + \frac{\gamma^2}{\sigma_M^2}\right)V''_{g,z} - q_{z,z+1}\left(V_{g,z+1} - V_{g,z}\right) = \\ k_z\left(1+\hat{\mu}_t\right)\left(V_K - 1\right) + k_z \gamma V'_K + \left(a_z k_z\right)\left(\left(1+\hat{\mu}_t\right)V_f + \gamma V'_f\right). \tag{IA.13}$$

Equations (IA.11) and (IA.12) are ordinary differential equations characterizing the functions $V_f\left(\hat{\mu}_t\right)$ and $V_K\left(\hat{\mu}_t\right)$, respectively. Equation (IA.13) is a system of ordinary differential equations characterizing the solutions of the functions $V_{g,z}\left(\hat{\mu}_t\right)$. Note that $V_{g,z}\left(\hat{\mu}_t\right) = V_{g,z+1}\left(\hat{\mu}_t\right) = 0$ when the firm is in the no-growth state $z = NG$. We solve the ordinary differential equations in equation (IA.11) through (IA.13) numerically using Chebyshev polynomial approximations.

II. Variable Definitions, Summary Statistics, and Stylized Facts from the Empirical Sample

The focus of the paper is on firms' valuation ratios and the underlying fundamental firm characteristics—the profitability of their assets and their growth opportunities. In this appendix, we describe the measures of firm characteristics that we use. We also present stylized facts from the empirical data sample.

The valuation ratio that we use to capture the value anomaly is *market-to-book assets* (*MB*), which is defined as the market value of assets divided by total book assets. The market value of assets equals the market value of equity lagged six months plus the book value of debt, where the latter is computed as total liabilities plus preferred stock minus the sum of deferred taxes and convertible debt. The empirical investment literature typically uses the book-to-market equity ratio to capture the value anomaly (e.g., Fama and French (1992)). We use *MB* because it measures firm (not just equity) value and thus more closely corresponds to investors' valuations in our model, which assumes that firms are 100% equity financed. In unreported analyses we find that using book-to-market equity instead produces similar results.

The market-to-book ratio in our model is a function of the growth rate and the profitability of the firm's assets. We follow Cooper, Gulen, and Schill (2008) and measure firm growth using *asset growth* (*AG*), which is the percentage change in book assets from the prior year to the current. To capture profitability, we use *operating profitability* (*OP*), which is the operating income before depreciation (total revenue minus cost of goods sold minus selling, general and administrative expenses) divided by total book assets. This measure is similar to that used by Fama and French (2015), but their measure of operating profits is net of interest expense and normalized by book equity. The reason for our modification is the same as in the previous paragraph – firms in our model are 100% equity financed and hence do not incur interest expenses. The results are again largely the same when we use the original Fama-French (2015) measure of operating profitability.[1]

Our sample includes common equity shares traded on the NYSE, AMEX, and NASDAQ over the 50-year period from July 1964 to June 2014. We exclude stocks in the smallest NYSE size decile, stocks with either negative book equity or price less than $5 at the time of portfolio construction, and financial firms (those with one-digit SIC code of six).

For each characteristic, we sort firms into five quintile portfolios. We label the top and bottom quintile portfolios according to the underlying characteristic (e.g., the value portfolio in a *MB* sort includes stocks that are in the bottom quintile of market-to-book assets). Portfolios are

[1] The literature documents profitability-based return predictability using other measures as well. Most notably, Novy-Marx (2013) analyzes strategies based on gross profitability (i.e., revenues minus cost of goods sold divided by assets). He finds that gross profitability by itself is a somewhat weaker predictor of returns in comparison to the Fama-French (2015) measure, but combining the value and gross profitability strategies achieves a very high Sharpe ratio due to the strong negative correlation between value and gross profitability.

formed at the end of June in each year, are value-weighted, and are rebalanced monthly.[2] In parts of our empirical analysis we also consider industry-adjusted characteristic portfolios, where stock returns are measured in excess of the 48 Fama-French industry portfolio returns. Monthly excess returns are computed by subtracting the one-month Treasury bill rate. Market-neutral returns are computed as the alpha plus the residual in monthly time-series CAPM regressions of the portfolio return on the value-weighted market return. While our data observations and analyses are at the monthly frequency, we report annualized volatilities and Sharpe ratios for ease of reference.

Table IA.I reports the median values of the three firm characteristics for each characteristic-sorted portfolio. As the table shows, all three characteristics exhibit substantial cross-sectional variation. Both profitability and asset growth are positively related to the market-to-book ratio. In particular, firms in the high profitability portfolio resemble growth firms, with market-to-book ratios exceeding two. There is also a positive association between profitability and asset growth, but the magnitude of this relationship is quite small relative to the cross-sectional variation that each characteristic exhibits.

Table IA.II reports return statistics for various investment strategies. In the first four rows of the table, we report the return standard deviations and the Sharpe ratios of four long-short portfolios: the market portfolio minus the risk-free asset, the low market-to-book minus the high market-to-book portfolio, the low asset growth minus the high asset growth portfolio, and the high profitability minus the low profitability portfolio. The Sharpe ratio of the market portfolio is 0.376, while the Sharpe ratios of the characteristic-sorted portfolios range from 0.281 to 0.322.

[2] Portfolios are value-weighted with respect to the equity market capitalizations of the stocks included in them. Equally-weighted portfolios, which are dominated by smaller stocks, may generate more extreme Sharpe ratios but may not be realistic because of the implicit monthly rebalancing.

While the Sharpe ratios of the characteristic-sorted portfolios reported above are quite high, it is not clear what they should be benchmarked against, as the underlying portfolios are exposed to market risk. Because our model abstracts from a market risk factor, we focus primarily on market-neutral portfolio returns, measured as the difference between the portfolio's raw return and its estimated beta times the market excess return. Under the CAPM, the Sharpe ratios of these market-neutral portfolios are zero in expectation. Since a portfolio's realized Sharpe ratio is a scaled *t*-statistic, one can calculate the probability of observing a realized Sharpe ratio under the CAPM from the relevant *t* distribution.[3] For instance, a right-tail *p*-value of 0.05 corresponds to a *t*-statistic of 1.647 with 50 years of data, which implies an annualized Sharpe ratio of 0.233 ($=1.647/\sqrt{50}$).

As shown in the middle panel in Table IA.II, taking out the market exposure substantially increases the Sharpe ratios of the three characteristic-sorted strategies. The market-neutral market-to-book, asset growth, and profitability strategies obtain Sharpe ratios of 0.385, 0.515, and 0.435, respectively. The corresponding *p*-values for these Sharpe ratios under the CAPM are 0.003, 0.0001, and 0.001. Thus, the historical Sharpe ratios of characteristic-sorted strategies have extremely low likelihoods under the null that the CAPM holds.

To see whether the documented return predictability patterns continue to hold within industries, we consider industry-adjusted (and also market-neutral) investment strategies as well. The results are reported in the right-side panel of Table IA.II. Taking out the industry returns further increases the Sharpe ratios of the market-to-book and asset growth strategies, and

[3] Specifically, a long-short portfolio's Sharpe ratio is the *t*-statistic on its mean return divided by the square root of *T*, which is the number of return observations. Our sample period is 50 years, or 600 months. The relevant *t* distribution therefore has 599 degrees of freedom.

substantially reduces their volatilities, but has a small impact on the profitability strategy. Thus, the main return patterns of interest hold when we sort within industries as well.

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Table IA.I
Median Firm Characteristics in the Historical Sample

The table reports median firm characteristics in characteristic-sorted portfolios. Market-to-book assets (*MB*) is the market value of assets (market value of equity lagged six months plus book value of debt, where the latter is computed as total liabilities plus preferred stocks minus the sum of deferred taxes and convertible debt) divided by total book assets. Asset growth (*AG*) is the percentage change in book assets from the prior year to the current. Operating profitability (*OP*) is operating income before depreciation (total revenue minus cost of goods sold minus selling, general, and administrative expenses) divided by total book assets. Portfolios are formed based on quintile values of the sorting characteristic.

	Portfolios Sorted on *OP*			Portfolios Sorted on *AG*			Portfolios Sorted on *MB*		
	OP	*AG*	*MB*	*OP*	*AG*	*MB*	*OP*	*AG*	*MB*
L	0.046	0.075	1.351	0.118	-0.036	1.165	0.109	0.048	0.896
2	0.110	0.082	1.103	0.139	0.042	1.185	0.129	0.075	1.112
3	0.145	0.089	1.228	0.154	0.100	1.333	0.149	0.099	1.376
4	0.186	0.109	1.517	0.167	0.186	1.572	0.178	0.131	1.874
H	0.262	0.149	2.323	0.151	0.483	1.806	0.209	0.208	3.424

Table IA.II
Portfolio Returns in the Historical Sample

The table reports mean returns, volatilities, and Sharpe ratios of long-short portfolios. All statistics are computed using monthly data and reported in annualized terms. Mean returns and volatilities are percentages. Market minus risk-free is the market portfolio return in excess of the risk-free rate. Low minus high market-to-book, low minus high asset growth, and high minus low profitability are based on the *MB*, *AG*, and *OP* sorts, respectively. All portfolios are value-weighted. Market-neutral returns are computed as the alpha plus the residual in monthly time-series CAPM regressions of the portfolio return on the value-weighted market return. Industry-adjusted portfolio returns are measured in excess of the 48 Fama-French industry portfolio returns.

	Raw Returns			Market-Neutral Returns			Industry-Adjusted Market-Neutral Returns		
Long-Short Portfolios	Mean Return	Volatility	Sharpe Ratio	Mean Return	Volatility	Sharpe Ratio	Mean Return	Volatility	Sharpe Ratio
Market minus Risk-Free	5.88	15.62	0.376	-	-	-	-	-	-
Low minus High Market-to-Book	4.07	13.74	0.297	5.17	13.42	0.385	3.03	5.76	0.526
Low minus High Asset Growth	3.76	11.66	0.322	5.51	10.69	0.515	3.63	5.96	0.610
High minus Low Profitability	3.71	13.22	0.281	5.40	12.43	0.435	3.84	9.04	0.425